# Representing Types through Image Schemas and Patterns

**Claudio Masolo** and **Emilio Sanfilippo**[1]

**Abstract.** We propose a formal framework to represent types intended as (intentional) complex properties characterised in terms of $(i)$ simple *qualities*, e.g., colors or weights; $(ii)$ *structural properties*, i.e., properties concerning the spatial distribution of qualities, e.g., being polka-dotted or being uniformly dense; and $(iii)$ *structured properties*, i.e., properties concerning the topological arrangement of the components of assemblies. Our framework is inspired by work done in cognitive science, in particular by Gärdenfors' theory of conceptual spaces, as well as by the notion of image schema, the latter being a theoretical construct to represent general patterns and the way they apply to heterogeneous cases. Hopefully, our approach can also contribute to the understanding of the notions of image schema and conceptual space.

## 1 Introduction

The formal representation of *types* is of fundamental relevance for conceptual modelling and knowledge representation purposes [17]. In the context of engineering design, for example, it is common to distinguish between *artefacts* as individual spatio-temporal objects, and *artefact types* as complex properties that classify individual artefacts. Take, e.g., John's and Mary's Fiat500 cars, namely two distinct objects. Classifying them as artefacts of the type Fiat500 means that they have a certain shape, weight, height and spatial arrangement of components that are characteristic of the Fiat500 type. Generalising from design, the distinction between types and their corresponding instances apply to disparate categories commonly used in knowledge representation, e.g., events, among others.

The modelling of types is challenging. Philosophers have been discussing about the nature of properties since the early days of philosophical disputes, e.g., whether they are mind-independent universals, or whether they reduce to human ways of categorising phenomena.[2] The latter view is at the core of cognitive studies according to which knowledge acquisition and perception are guided from (or give rise to) some sort of *conceptual structures* [4, 20].

Departing from a metaphysical stance, we propose an approach for the representation of types grounded on cognitive theories. More specifically, we shall focus on the *compound* nature of types, namely, the fact that they are characterised in terms of properties concerning $(i)$ qualitative aspects (*qualities*), $(ii)$ the distribution of such qualities across space and/or time (*structural properties*), or $(iii)$ other types each one classifying the component of a topologically structured assembly (*structured properties*).

The proposed framework is based on a modified version of Gärdenfors' theory of conceptual spaces [4], the latter being well-suited to represent types as multi-dimensional properties, therefore as being composed of other properties. The framework is also inspired by studies on *image schemas* (e.g., [9, 11]). Even though the latter are challenging to be captured in detail, there is a relative consensus in understanding them as sorts of (very general) patterns. Johnson claims that an image schema is "a dynamic pattern that functions somewhat like the abstract structure of an image, and thereby connects up a vast range of different experiences that manifest this same recurring structure" [9, p.2]. According to Langacker, image schemas are "schematized patterns of activity abstracted from everyday bodily experience, especially pertaining to vision, space, motion, and force" [11, p.42]. Recent approaches devoted to formal aspects describe image schemas as "patterns abstracting from spatio-temporal experiences" [10, p.155], or "mental patterns [that] may be combined with each other to generate more complex structures" [7, p.21]. We find image schemas useful theoretical constructs to represent recurrent configurations and the way they apply, in different contexts and with different modalities, to heterogeneous cases. The paper is not a contribution to clarifying or formalising conceptual spaces or image schemas; it rather proposes a way to re-elaborate some approaches in cognitive studies to deal with some knowledge representation issues, as done, e.g., in [10]. Hopefully, however, we can also shed some light on the very notions (and limitations) of conceptual space and image schema.

The paper is structured as follows. In Sect. 2 we introduce the main motivations behind our work. Sect. 3 presents the overall framework for the representation of qualities. Sect. 4 introduces the machinery to represent image schemas, which are used in Sect. 5 to represent patterns of qualities. Sect. 6 presents how patterns may be used to classify the structural properties of objects according to either time or space. Finally, Sect. 7 shows how structured properties and complex types can be represented in the proposed framework.

## 2 Properties: Types and qualities

In conceptual modelling and knowledge engineering, one commonly distinguishes between *individuals* and *properties*, including a predication or classification relationship holding between them. Recalling the example mentioned in the introduction, John's and Mary's cars are two individuals of the same type Fiat500, namely, they are both classified by the same (complex) property.

Properties bear an *intensional* nature, i.e., differently from sets, they do not reduce to their members (extensions). As a consequence, different properties can classify the same individuals, e.g., *being triangular* and *being trilateral*.[3] We distinguish two main kinds of

[1] Laboratory for Applied Ontology, ISTC-CNR, Trento, Italy, email: {masolo,sanfilippo}@loa.istc.cnr.it

[2] Roughly, this is the distinction between realists and nominalists concerning the metaphysics of properties.

[3] The intensional characterisation of properties is relevant for different domain-specific scenarios. For example, in the context of design, artefact types (properties) are not usually specified by listing their instances but by sets of constraints on relevant characteristics.

properties, namely *types* and *qualities*. Types are *compounds* of simpler properties. Among these properties there are the qualities that are the simplest properties, i.e., they are not further decomposable.[4] For example, the type Fiat500 is composed by qualities like colour, shape and weight, among others, which constraint the holistic properties of Fiat500 cars.

Some types classify assemblies with a specific spatial structure. Fiat500 cars, for instance, consist of disparate components properly arranged, e.g., bodywork, engine and wheels. From this perspective, types do not reduce to compounds of qualities; they are characterised also in terms of what Armstrong calls *structured properties*. Similarly, Fiorini and colleagues [21, 22] claim that objects recognition and categorisation is not only based on the global (aka holistic) features that objects manifest, but it is also achieved by identifying objects' structures in terms of their parts and the way in which parts are arranged. A theory of cognition has, therefore, to deal with conceptual structures about both holistic and structural information.

From a representational perspective, we propose to model the type/quality distinction following the *similarity space theory of concepts* [20] and, in particular, the theory of *conceptual spaces* introduced by Gärdenfors [4]. The latter theory provides the basic framework to convey the intensional nature of properties, to characterise the complex nature of types in opposition to qualities, as well as to compare different types on the basis of their degree of similarity.

Gärdenfors [4] represents *concepts*—e.g., car, apple, etc.—as regions in a *space* obtained by composing a given number of *domains*—e.g., colour, taste, etc. The main reason to decompose a space into domains is the assumption that the properties in the domains may be attributed to individuals *independently* of the properties in other domains; e.g., the weight of an individual is independent of its temperature or colour.[5] Additionally, conceptual spaces have the peculiarity to be endowed with a distance relation representing degrees of similarity: the closer are the properties (within a domain), the more similar are the individuals that exhibit such properties.

The representation of structured properties has proven challenging in conceptual spaces. In some studies, Fiorini and colleagues [21, 22] propose a theory of part-whole relations with the purpose of grounding structural relations in cognition. The authors extend Gärdenfors' theory with *structural spaces*, which allow for the representation of the arrangements of the parts within a whole according to certain configurations.[6] However, the proposed framework has a mathematical and combinatorial nature, whereas the modality according to which the general patterns apply to specific cases is missing. Furthermore, the spatial structure of objects is not only relevant to represent how types (and the entities they classify) relate in a part-whole manner. Indeed, it also plays a fundamental role in cases in which types are characterised by *distributional* properties [18]. Consider, e.g., a car-type whose bodywork-type is characterised by a colour lozenge pattern distributed across the overall instances of the bodywork-type. In a design perspective, there are no components of the bodywork-type, each being coloured in a certain way. We consider these properties, called *structural* properties, in Sect. 6. It remains unclear how Fiorini and colleagues would approach this case.

[4] The distinction between types and qualities is reminiscent of the one, between *classes* and *attributes* assumed in the field of conceptual modelling, as, e.g., in the UML class-diagram [17]. Furthermore, some qualities, e.g., colours, may be compound, too [4]; we do not consider this aspect here.

[5] The domains are not totally independent, i.e., they may be *correlated*.

[6] A point in a structural space represents also the particular configuration of the parts of an object [21].

## 3 Qualities

We start by introducing two disjoint kinds of properties: $\mathrm{QT}(x)$ stands for "$x$ is a *quality*", and $\mathrm{TY}(x)$ stands for "$x$ is a *type*". This distinction roughly reflects the one between simple and compound properties, i.e., in terms of conceptual spaces, the distinction between properties (in a domain) and concepts. Qualities are partitioned into a finite number $\eta$ of *domains* $\mathrm{D}_i$, e.g., the domains of colour, shape and weight. Axiom (Ax1) guarantees that all the qualities belong to at least one domain, while (Ax2) assures that the domains are disjoint.

**Ax1** $\mathrm{QT}(x) \leftrightarrow \bigvee_{i=1}^{\eta} \mathrm{D}_i(x)$

**Ax2** $\bigwedge_{i \neq j=1}^{\eta} (\mathrm{D}_i(x) \rightarrow \neg \mathrm{D}_j(x))$

For instance, $\mathrm{D}_{\mathrm{colour}}(\mathtt{red})$ states that the quality *being red* belongs to the colour-domain, while $\mathrm{D}_{\mathrm{weight}}(\mathtt{1506kg})$ that the quality *being 1506kg heavy* belongs to the weight-domain.[7]

Every quality domain has a *top-quality*, i.e., a quality that subsumes all the qualities in the same domain (Ax3). The top-quality of a domain is noted with the name of the domain, e.g., the top-quality of $\mathrm{D}_{\mathrm{weight}}$ is noted with `weight`.

**Ax3** $\bigwedge_{i=1}^{\eta} \exists d \forall q (\mathrm{D}_i(q) \rightarrow q \sqsubseteq d)$

We include *time* and *space* among the quality domains represented, respectively, with TM and SP (rather than $\mathrm{D}_{\mathrm{time}}$ and $\mathrm{D}_{\mathrm{space}}$). This move is motivated by the fact that time and space are structured similarly to other domains. However, we attribute them a peculiar role in the classification relation, since they *qualify* the classification (see Sect. 3.1). Recall that some philosophers advocate the ontological primacy of time and space. In particular, spatial relations are claimed to not be reducible to property-grounded relations [8]. We do not however enter into this discussion; our choice is just aimed at simplifying the formalisation. Proper qualities, noted as PQT and simply called qualities, are the qualities that are neither times nor spaces.

Domains contain comparable qualities. More generally, one can think of domains as sets of intensionally interlinked qualities. We introduce a (intensional) *subsumption* relation between qualities belonging to the same domain; see (Ax4), where $x \sqsubseteq y$ stands for "the quality $x$ *is intensionally subsumed* by the quality $y$". $\mathrm{D}_i$-qualities cannot be subsumed by $\mathrm{D}_j$-qualities (with $i \neq j$). For example, `scarlet` cannot be subsumed by any quality in $\mathrm{D}_{\mathrm{weight}}$; it could be rather subsumed by `red`, i.e., $\mathtt{scarlet} \sqsubseteq \mathtt{red}$.

**Ax4** $x \sqsubseteq y \rightarrow \bigvee_{i=1}^{\eta} (\mathrm{D}_i(x) \wedge \mathrm{D}_i(y))$

Proper subsumption $\sqsubset$ is defined in (Df1). Formally, $\sqsubseteq$ is a discrete and atomic partial order: it is reflexive, antisymmetric, transitive, discrete, and atomic, i.e., every quality subsumes an *atomic* quality, a quality that does not properly subsume any other quality (Df2). Ontologically, atomic qualities represent how the world is; epistemologically, they represent the maximal resolving power one disposes of, e.g., the finest resolution of measurement devices.

**Df1** $x \sqsubset y \triangleq x \sqsubseteq y \wedge \neg y \sqsubseteq x$

**Df2** $\mathrm{AT}_{\mathrm{QT}}(x) \triangleq \mathrm{QT}(x) \wedge \neg \exists y (y \sqsubset x)$

Quality domains may be structured according to further relations. Images schemas often refer to topological or order structures. We avoid to introduce a full metrics, as usually done for conceptual spaces, by assuming all domains to be endowed with at least the *con-*

[7] We note individual constants using the `typewriter` font.

*nection* relation $\bowtie$.[8] *Ordered* domains, e.g., $\text{TM}$, $\text{D}_{\text{weight}}$, or $\text{D}_{\text{length}}$, are also (linearly) ordered by the *precedence* relation $\prec$. Both connection and precedence hold only among qualities in the same domain (Ax5)-(Ax6).

**Ax5** $x \bowtie y \rightarrow \bigvee_{i=1}^{\eta}(\text{D}_i(x) \wedge \text{D}_i(y))$

**Ax6** $x \prec y \rightarrow \bigvee_{i=1}^{\eta}(\text{D}_i(x) \wedge \text{D}_i(y))$

Notice that qualities may subsume different (atomic) qualities, therefore, times, weights, colors, shapes, etc. are more similar to regions rather than to points, i.e., $\sqsubseteq$ could be interpreted mereologically. In this perspective, precedence ($\prec$) could be seen as the disjunction of the relations of *meet* and *before* introduced in [1], while the RCC calculus [23] may be considered for connection ($\bowtie$). The detailed axiomatic treatment of $\prec$ and $\bowtie$ is not relevant for our purposes; we want just to highlight the possibility to have structural relations (in addition to subsumption) that are defined in several (or all) quality domains. As we will see, this aspect is fundamental to characterise image schemas and patterns.

### 3.1 Classification

We analyse hereby how qualities can be attributed to individuals. Since qualities are in the domain of quantification, the standard predication mechanism of FOL cannot be adopted. It is thus necessary to introduce a new primitive relation, named *classification*, and to establish under what conditions individuals are classified by qualities. Here we focus on the classification of *physical objects* (noted $\text{OB}$), aka *continuants* or *endurants*, that are in, and can change through, space and time, e.g., my car, Barack Obama, the earth, etc. The framework can be however applied, with minimal tuning, also to *events*, aka *occurrents*, *perdurants*.

We start from a *local* notion of *direct classification* under qualities: $\texttt{dCF}_{\text{QT}}(q, x, s, t)$ stands for "the (proper) quality $q$ directly classifies the object $x$ *as it is at space $s$ and time $t$*".[9] Given the possibility for objects to change through both time and space, the classification is spatio-temporally *qualified*.[10] For instance, John can decide to paint his red volleyball half in orange and half in blue.

**Ax7** $\texttt{dCF}_{\text{QT}}(q, x, s, t) \rightarrow \text{PQT}(q) \wedge \text{OB}(x) \wedge \text{SP}(s) \wedge \text{TM}(t)$

One could interpret $\texttt{dCF}_{\text{QT}}$ taking a *mereological* stance: when $\texttt{CF}_{\text{QT}}(q, x, s, t)$ holds, the entity that has the quality $q$ is not $x$ but a different object, namely the part of $x$ that, at $t$, is (exactly) located at space $s$. We assume a weaker position that does not equate the resolution of space with the one of objects, i.e., atomic objects can be located in non-atomic spaces. We come back to this point in Sect. 7.

By relying on subsumption, (Df3) defines the relation of *local indirect classification*: $q$ indirectly classifies $x$ as it is at space $s$ and time $t$ when $x$ is locally classified under a quality that is (properly) subsumed by $q$. For example, if $\texttt{crimson} \sqsubset \texttt{red}$ and $\texttt{dCF}_{\text{QT}}(\texttt{crimson}, \texttt{jball}, \texttt{s}, \texttt{t})$, then $\texttt{iCF}_{\text{QT}}(\texttt{red}, \texttt{jball}, \texttt{s}, \texttt{t})$ ($\texttt{jball}$ stands for John's volleyball).

**Df3** $\texttt{iCF}_{\text{QT}}(q, x, s, t) \triangleq \exists z(\texttt{dCF}_{\text{QT}}(z, x, s, t) \wedge z \sqsubset q)$

(Df3) captures a disjunctive reading of classification: it is enough to be directly classified under *one* of the qualities that are subsumed by $q$ to be indirectly classified under $q$. For instance, suppose $\texttt{crimson} \sqsubset \texttt{red}$ and $\texttt{scarlet} \sqsubset \texttt{red}$. Both scarlet and crimson entities are indirectly classified under $\texttt{red}$. The difference between direct and indirect classification is *cognitively* and *empirically* relevant. Direct classification is the result of a direct observation or sensation while indirect classification relies on an abstraction process. Objects can be thus directly classified under non-atomic qualities, i.e., it is possible to have partial or general information due to the resolution at which the world is accessed. It is possible, for instance, to know that John's volleyball is red without knowing its exact shade. This means that direct and indirect classification are not mutually exclusive. For instance, in an empirical context, it is possible to have measurements about the same object taken with devices that have different resolutions. This is a departure from the Gärdenfors' theory of conceptual spaces where an object is always represented by one single point in a space.[11] The general relation of local classification abstracts from the direct vs. indirect distinction (Df4).

**Df4** $\texttt{CF}_{\text{QT}}(q, x, s, t) \triangleq \texttt{dCF}_{\text{QT}}(q, x, s, t) \vee \texttt{iCF}_{\text{QT}}(q, x, s, t)$

To clarify what the qualification "at space $s$ and time $t$" means we introduce the primitive relation of *being present*: $\texttt{PRE}(x, s, t)$ stands for "the object $x$ *is present* at the spatio-temporal region identified by space $s$ and time $t$", i.e., "at $t$, $x$ occupies $s$" (Ax8).[12] As said, we focus on spatio-temporal objects, i.e., objects that are present both in time and space (Ax9). The local classification (at $s$ and $t$) of an object requires its presence (at $s$ and $t$), see (Ax10).

**Ax8** $\texttt{PRE}(x, s, t) \rightarrow \text{OB}(x) \wedge \text{SP}(s) \wedge \text{TM}(t)$

**Ax9** $\text{OB}(x) \rightarrow \exists st(\texttt{PRE}(x, s, t))$

**Ax10** $\texttt{CF}_{\text{QT}}(q, x, s, t) \rightarrow \texttt{PRE}(x, s, t)$

First note that being present at $s$ and $t$ is not equivalent to being present at $s$ *and* being present at $t$. Second, and more importantly, being present at $s$ and $t$ is not a disjunctive abstraction from being present at some atomic subregions of $s$ and $t$.[13] For $\texttt{PRE}$ we have a sort of conjunctive reading: being present at $s$ and $t$ requires to be present at every subregion (Ax11).[14]

**Ax11** $\texttt{PRE}(x, s, t) \wedge r \sqsubset s \wedge u \sqsubset t \rightarrow \texttt{PRE}(x, r, u)$

Spatial location (at $t$) is defined in (Df5).[15] (Df6) introduces the classification of an object as it is at its whole spatial location, i.e., it determines the *global* (aka *holistic*) qualities of an object.

**Df5** $\text{L}_{\text{SP}}(x, s, t) \triangleq \texttt{PRE}(x, s, t) \wedge \neg\exists r(\texttt{PRE}(x, r, t) \wedge s \sqsubset r)$

**Df6** $\texttt{CF}_{\text{QT}}(q, x, t) \triangleq \exists s(\text{L}_{\text{SP}}(x, s, t) \wedge \texttt{CF}_{\text{QT}}(q, x, s, t))$

Extensional relations between qualities can be introduced by relying on the global $\texttt{CF}_{\text{QT}}$. By (Df7), $x \subseteq y$ (read as "$x$ is (extensionally) *included* in $y$") holds when every entity globally classified at

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[8] Some domains could have alternative topological organisations, e.g., the colour spindle or the RGB colour wheel, see [4]. In the following we assume domains to have a unique topological organisation.

[9] Local classification is much more informative than holistic classification.

[10] Classification could also be qualified with proper qualities. For instance, $\texttt{dCF}_{\text{QT}}(\texttt{10kg}, x, \texttt{red}, t)$ would be read as "$x$ weights $\texttt{10kg}$ as it is at colour $\texttt{red}$ and time $t$". We leave this interesting extension to future work.

[11] See [5, 14] for some criticisms about this assumption.

[12] Here we assume a sort of container-like notion of space. One could substitute spaces with *places*, spatial entities that are more abstract than spaces and that can be defined relatively to physical objects, e.g., the top, the interior, etc., see [2]. Similarly for times.

[13] As for the characterisation of types (see Sect. 7), approximate localisation in space-time can be represented by means of rough sets (see [14]).

[14] One could think to introduce a *conjunctive* classification primitive $\texttt{cCF}$ that behaves similarly to $\texttt{PRE}$. For instance, $\texttt{cCF}(\texttt{red}, x, s, t)$ would imply a classification by all the color-qualities subsumed by $\texttt{red}$, e.g., $\texttt{cCF}(\texttt{scarlet}, x, s, t)$, $\texttt{cCF}(\texttt{crimson}, x, s, t)$, etc. In Sect. 6 we introduce structural properties, that, in our opinion, are stronger tools to characterise properties that require an object to have different qualities of the same kind.

[15] The location is unique only when specific constraints on $\text{SP}$ hold.

any time by the quality $x$ is also globally classified, at the same time, by $y$. For example, assume that all the red entities in the domain of quantification also weigh 1kg, then we have $\mathtt{red} \subseteq \mathtt{1kg}$. (Th1) can be trivially proved. However, the vice versa does not hold. This makes explicit the intensional nature of $\sqsubseteq$. Furthermore, different qualities may have the same extensions, that is, $x \subseteq y \wedge y \subseteq x$ does not imply $x = y$. Finally, note that empty-qualities, i.e., qualities that classify no object, are included in, but not subsumed by, all the qualities.

**Df7** $x \subseteq y \triangleq \mathrm{PQT}(x) \wedge \forall zt(\mathtt{CF}_{\mathrm{QT}}(x,z,t) \rightarrow \mathtt{CF}_{\mathrm{QT}}(y,z,t))$

**Th1** $x \sqsubseteq y \rightarrow x \subseteq y$

## 4 Image schemas

As outlined in [16], images schemas are very abstract spatial structures with qualitative (topological) characteristics, i.e., they are not precisely characterised in terms of (geometric) magnitude or shape. This makes image schemas "highly flexible preconceptual and primitive patterns" [16, p.217] that can be instantiated in different contexts. They are 'malleable' enough "to fit many similar, but different, situations that manifest a recurring underlying structure" [9, p.30]. Our idea is to capture these abstract structures by relying on the qualitative relations of connection ($\bowtie$) and order ($\prec$) defined in the quality domains, i.e., intuitively, an image schema classifies tuples of qualities satisfying structural constraints expressible via $\bowtie$ and $\prec$.

Technically, image schemas are represented by *higher-level qualities*, qualities of tuples of homogeneous qualities. As we will see, this approach generalises the one in [4] where 'patterns' of locations of objects along a quality domain are represented as higher-level properties (sets of tuples in *product spaces*).

Tuples, represented by TU, are introduced into the domain of quantification. They are disjoint from objects, as well as from qualities and types. Tuple variables and tuple constants are noted with $\bar{u}, \bar{v}$, etc. Tuples are formally characterised by following the strategy usually exploited to reify relationships in FOL [6]. To cope with a first-order formalisation, tuples are assumed to have a maximal length $\ell$. We thus introduce $\ell$ primitive binary relations $\multimap_{\mathtt{i}}$, where $x \multimap_{\mathtt{i}} \bar{u}$ refers to the $i$th element $x$ of the tuple $\bar{u}$, see (Ax12) and (Ax13). Axioms (Ax14) and (Ax15) assure that tuples have at least two elements, and when tuples have the $i$th element, they also have all the previous elements. (Ax16) establishes the identity criterion for tuples, i.e., two tuples are identical if they have the same elements in the same order. It is easy to prove the unicity of the tuple $\bar{u}$ such that $x_1 \ldots x_n \multimap \bar{u}$ (with $n \leq \ell$), see (Df8), that is noted $\langle x_1, \ldots, x_n \rangle$. As expected, the same element can appear in different positions of the same tuple, e.g., $\langle \mathtt{a}, \mathtt{b} \rangle$, $\langle \mathtt{a}, \mathtt{b}, \mathtt{a}, \mathtt{a} \rangle$ or $\langle \mathtt{a}, \mathtt{a} \rangle$ are all valid and different tuples. (Df9) establishes when two tuples have the same length, while $\mathrm{TU}_{\neq}$ identifies the tuples with all different elements (Df10). The subsumption relation between tuples is defined in (Df11).

**Df8** $x_1 \ldots x_n \multimap \bar{u} \triangleq \bigwedge_{\mathtt{i}=1}^{n}(x_{\mathtt{i}} \multimap_{\mathtt{i}} \bar{u}) \wedge \bigwedge_{\mathtt{i}=n+1}^{\ell} \neg \exists x(x \multimap_{\mathtt{i}} \bar{u})$

**Df9** $\bar{u} \equiv_{\mathrm{L}} \bar{v} \triangleq \mathrm{TU}(\bar{u}) \wedge \bigwedge_{\mathtt{i}=1}^{\ell}(\exists x(x \multimap_{\mathtt{i}} \bar{u}) \leftrightarrow \exists y(y \multimap_{\mathtt{i}} \bar{v}))$

**Df10** $\mathrm{TU}_{\neq}(\bar{u}) \triangleq \mathrm{TU}(\bar{u}) \wedge \bigwedge_{\mathtt{i} \neq \mathtt{j}=1}^{\ell} \forall xy(x \multimap_{\mathtt{i}} \bar{u} \wedge y \multimap_{\mathtt{j}} \bar{u} \rightarrow x \neq y)$

**Df11** $\bar{u} \sqsubseteq_{\mathrm{TU}} \bar{v} \triangleq \bar{u} \equiv_{\mathrm{L}} \bar{v} \wedge \bigwedge_{\mathtt{i}=1}^{\ell} \forall xy(x \multimap_{\mathtt{i}} \bar{u} \wedge y \multimap_{\mathtt{i}} \bar{v} \rightarrow x \sqsubseteq y)$

**Ax12** $\bigwedge_{\mathtt{i}=1}^{\ell} \forall x\bar{u}(x \multimap_{\mathtt{i}} \bar{u} \rightarrow \mathrm{TU}(\bar{u}))$

**Ax13** $\bigwedge_{\mathtt{i}=1}^{\ell} \forall xy\bar{u}(x \multimap_{\mathtt{i}} \bar{u} \wedge y \multimap_{\mathtt{i}} \bar{u} \rightarrow x = y)$

**Ax14** $\mathrm{TU}(\bar{u}) \rightarrow \bigvee_{\mathtt{i} \neq \mathtt{j}=1}^{\ell} \exists xy(x \multimap_{\mathtt{i}} \bar{u} \wedge y \multimap_{\mathtt{j}} \bar{u})$

**Ax15** $\mathrm{TU}(\bar{u}) \rightarrow \bigwedge_{\mathtt{i}=2}^{\ell}(\exists x(x \multimap_{\mathtt{i}} \bar{u}) \rightarrow \exists y(y \multimap_{\mathtt{i-1}} \bar{u}))$

**Ax16** $\bar{u} = \bar{v} \leftrightarrow \mathrm{TU}(\bar{u}) \wedge \bigwedge_{\mathtt{i}=1}^{\ell} \forall x(x \multimap_{\mathtt{i}} \bar{u} \leftrightarrow x \multimap_{\mathtt{i}} \bar{v})$

For our purposes, it is important to individuate tuples of qualities belonging to the same domain, i.e., $[\mathrm{D}_i]\mathrm{TU}$-tuples (see (Df13) where $\multimap\!\!\diamond$ is defined in (Df12)), e.g., $[\mathrm{TM}]\mathrm{TU}$-, $[\mathrm{SP}]\mathrm{TU}$-, or $[\mathrm{D}_{\mathrm{color}}]\mathrm{TU}$-tuples. DTU abstracts from the specific quality domain (Df14). Hedblom and colleagues [7] define *paths* as "collection[s] of two or more sites, which are connected by successor relationships" [7, p.27]. DTU-tuples can be seen as paths where sites are qualities (locations of objects) in a domain and the successor relation is captured by the order of the elements in the tuple. For this reason, we will use the terms path and DTU-tuple interchangeably.

**Df12** $x \multimap\!\!\diamond\, \bar{u} \triangleq \bigvee_{\mathtt{i}=1}^{\ell} x \multimap_{\mathtt{i}} \bar{u}$

**Df13** $[\mathrm{D}_i]\mathrm{TU}(\bar{u}) \triangleq \mathrm{TU}(\bar{u}) \wedge \forall x(x \multimap\!\!\diamond\, \bar{u} \rightarrow \mathrm{D}_i(x))$

**Df14** $\mathrm{DTU}(\bar{u}) \triangleq \bigvee_{i=1}^{\eta}[\mathrm{D}_i]\mathrm{TU}(\bar{u})$

*Image schemas*, represented by IM, are disjoint from all the other kinds of entities, QT included, i.e., QT collects only the qualities that directly classify objects. Image schema variables (constants) are noted with $\iota, \sigma$, etc. (using small caps). We assume (tuples of) qualities to be static, i.e., the direct classification $\mathtt{dCF}_{\mathrm{IM}}$ between image schemas and tuples of qualities does not need to be spatio-temporally qualified. In addition, we assume image schemas to classify only homogeneous qualities (Ax17).

**Ax17** $\mathtt{dCF}_{\mathrm{IM}}(\sigma, \bar{u}) \rightarrow \mathrm{IM}(\sigma) \wedge \mathrm{DTU}(\bar{u})$

Intuitively, image schemas capture general structural constraints. This idea can be formalised by means of constraints like (Ax18)-(Ax20), which characterise, respectively, the image schema 3MC of triples of mutually connected qualities, the image schema PWC of pairwise connected qualities, and the image schema INC of increasing qualities. These constraints do not refer to specific quality domains, they involve only shared structural relations. Image schemas are thus cross-domains. For example, $\langle \mathtt{1kg}, \mathtt{2kg}, \mathtt{4kg} \rangle$ and $\langle \mathtt{1m}, \mathtt{7m} \rangle$ are both instances, with different length, of INC.

**Ax18** $\mathtt{dCF}_{\mathrm{IM}}(\mathrm{3MC}, \bar{u}) \leftrightarrow \exists xyz(xyz \multimap \bar{u} \wedge x \bowtie y \wedge y \bowtie z \wedge z \bowtie x)$

**Ax19** $\mathtt{dCF}_{\mathrm{IM}}(\mathrm{PWC}, \bar{u}) \leftrightarrow \bigwedge_{\mathtt{i}=1}^{\ell-1} \forall xy(x \multimap_{\mathtt{i}} \bar{u} \wedge y \multimap_{\mathtt{i+1}} \bar{u} \rightarrow x \bowtie y)$

**Ax20** $\mathtt{dCF}_{\mathrm{IM}}(\mathrm{INC}, \bar{u}) \leftrightarrow \bigwedge_{\mathtt{i}=1}^{\ell-1} \forall xy(x \multimap_{\mathtt{i}} \bar{u} \wedge y \multimap_{\mathtt{i+1}} \bar{u} \rightarrow x \prec y)$

Note that the form of these constraints—for instance, the fact that they involve only the $\prec$ and $\bowtie$ primitives—can be regulated only at the meta-level. The structural nature of image schemas is thus only poorly captured by our FOL framework. A second problem concerns the intensional nature of image schemas. Gärdenfors builds patterns in a purely mathematical manner as subsets of product spaces. Accordingly, Gärdenfors' patterns have a purely *extensional* nature. Vice versa one could think, for instance, that the 3-long patterns *being equidistant* and *forming the same angles* defined on triples of (punctual) spaces are co-extensional but different. In principle, our framework is compatible with this intensional stance. However, the formulas that characterise the *being equidistant* and *forming the same angles* patterns are (in Euclidean geometry) logically equivalent. One should then assume the *way* patterns are characterised through axioms to impact the identity of patterns. The proper characterisation of the intension of image schemas is left for future work.

Our image schemas are similar to predicates that apply to tuples of qualities. In this sense they are close to the patterns of Gärdenfors [4]. There are however some important differences that show that our image schemas are more abstract and more flexible than patterns. First, in [4], all the instances of a pattern are tuples of a product space obtained from a given domain. For instance, following [4], `lighter` and `shorter` are two different patterns. Vice versa, the image schema INC is flexible enough to apply both to weights and lengths. Sect. 5 shows how the relations *being lighter than* and *being shorter than* are obtainable by applying the image schema INC to weights and lengths, respectively. Second, according to [4], all tuples instantiating a pattern have the same length. We have already shown that this does not hold for image schemas, which may be indeed *cyclic* (see, e.g., [7]) or—using the terminology of Galton [3]—*open*, i.e., without a pre-established length.

## 5 Patterns

We introduce hereby *patterns* that result from the instantiation of image schemas in given contexts. Intuitively, a context localises an image schema in the sense that it constrains the qualities the schema can apply to. We represent contexts by DTU-tuples and patterns (noted with PT) by couples $\langle\sigma,\bar{c}\rangle$ where $\mathrm{IM}(\sigma)$ and $\mathrm{DTU}(\bar{c})$, i.e., they are image schemas together with contexts of application. Pattern variables and constants are noted $\hat{p},\hat{q}$, etc. The context $\bar{c}$ constrains the schema $\sigma$ in the sense that it filters out all the tuples classified by $\sigma$ that are not subsumed by $\bar{c}$, see (Df15).[16] Furthermore, $\bar{c}$ sets the length of the pattern, i.e., by (Df11), all the instances of $\langle\sigma,\bar{c}\rangle$ have the length of $\bar{c}$. Patterns have then a fixed lenght.

**Df15** $\mathtt{dCF}_{\mathrm{PT}}(\langle\sigma,\bar{c}\rangle,\bar{u}) \triangleq \mathrm{PT}(\langle\sigma,\bar{c}\rangle) \wedge \mathtt{dCF}_{\mathrm{IM}}(\sigma,\bar{u}) \wedge \bar{u} \sqsubseteq_{\mathrm{TU}} \bar{c}$

Following Gärdenfors, our patterns can be used to classify tuples of *objects*, i.e., to represent *internal relations* among objects (Df16).[17] For instance, $\langle\mathrm{INC},\langle\mathtt{weight},\mathtt{weight}\rangle\rangle$ represents *being lighter than*, while $\langle\mathrm{INC},\langle\mathtt{lenght},\mathtt{lenght}\rangle\rangle$ represents *being shorter than*.

**Df16** $\mathtt{dCF}(\langle\sigma,\bar{c}\rangle,\bar{x},t) \triangleq \mathrm{PT}(\langle\sigma,\bar{c}\rangle) \wedge \bar{c} \equiv_{\mathrm{L}} \bar{x} \wedge \exists\bar{u}(\mathtt{dCF}_{\mathrm{PT}}(\langle\sigma,\bar{c}\rangle,\bar{u}) \wedge \bigwedge_{\mathtt{i}=1}^{\ell} \forall ux(u \multimap_{\mathtt{i}} \bar{u} \wedge x \multimap_{\mathtt{i}} \bar{x} \to \mathtt{CF}_{\mathrm{QT}}(u,x,t))$

The subsumption relation between patterns is defined in (Df17). Note that (Df17) requires the identity of image schemas, i.e., only instances of the same schema are comparable. For instance, by requiring tuples to have specific color-qualities, $\langle\mathrm{PWC},\langle\mathtt{red},\mathtt{blue},\mathtt{red}\rangle\rangle$ is subsumed by $\langle\mathrm{PWC},\langle\mathtt{color},\mathtt{color},\mathtt{color}\rangle\rangle$.

**Df17** $\langle\sigma_1,\bar{c}_1\rangle \sqsubseteq_{\mathrm{PT}} \langle\sigma_2,\bar{c}_2\rangle \triangleq \sigma_1 = \sigma_2 \wedge \bar{c}_1 \sqsubseteq_{\mathrm{TU}} \bar{c}_2$

To sum up, a pattern $\langle\sigma,\bar{c}\rangle$ captures the same structure of an image schema $\sigma$ but it is less abstract than $\sigma$ because $(i)$ it has a fixed-length (set by $\bar{c}$); and $(ii)$ it focuses on the (DTU-)tuples subsumed by $\bar{c}$.

## 6 Structural properties

With the technical and conceptual machinery previously introduced, we now approach the representation of *structural properties* of objects, i.e., properties that take into account the *way* a pattern of qualities is exhibited by an object. Here we consider temporal and spatial ways (aka modalities) of exhibiting patterns. Roughly, our idea is that an object has a structural quality when it exhibits a pattern of qualities temporally or spatially arranged according to a second pattern. Structural properties, noted with SPR, are represented by couples of patterns $\langle\hat{q},\hat{a}\rangle$, where $\hat{a}$ constrains the (temporal or spatial) arrangement of the qualities constrained by $\hat{q}$. We assume the context of $\hat{a}$ to have the same length of the context of $\hat{q}$ and to be $(i)$ a $[\mathrm{TM}]\mathrm{TU}$-tuple or $(ii)$ a $[\mathrm{SP}]\mathrm{TU}$-tuple. Structural properties with a temporal arrangement (case $(i)$) are called *historical properties* (see [15]), while the ones with a spatial arrangement (case $(ii)$) are called *distributional properties* (see [18]). (Df18) defines subsumption between structural properties.

**Df18** $\langle\hat{q}_1,\hat{a}_1\rangle \sqsubseteq_{\mathrm{SPR}} \langle\hat{q}_2,\hat{a}_2\rangle \triangleq \mathrm{SPR}(\langle\hat{q}_1,\hat{a}_1\rangle) \wedge \mathrm{SPR}(\langle\hat{q}_2,\hat{a}_2\rangle) \wedge \hat{q}_1 \sqsubseteq_{\mathrm{PT}} \hat{q}_2 \wedge \hat{a}_1 \sqsubseteq_{\mathrm{PT}} \hat{a}_2$

The classification of an object under an historical property is defined in (Df19): at $t$, the object $x$ is classified by $\langle\hat{q},\hat{a}\rangle$ (where $\hat{a}$ is a temporal arrangement) when, *before* $t$, following the *time* $\hat{a}$-path $\bar{a}$, the object $x$ *holistically* follows the *quality* $\hat{q}$-path $\bar{q}$.[18] According to (Df19), *historical* properties hold in virtue of the temporal distribution of the qualities the classified object had in its past history.[19]

**Df19** $\mathtt{tCF}(\langle\hat{q},\hat{a}\rangle,x,t) \triangleq \mathrm{SPR}(\langle\hat{q},\hat{a}\rangle) \wedge [\mathrm{TM}]\mathrm{PT}(\hat{a}) \wedge \exists\bar{q}\bar{a}(\mathtt{dCF}_{\mathrm{PT}}(\hat{q},\bar{q}) \wedge \mathtt{dCF}_{\mathrm{PT}}(\hat{a},\bar{a}) \wedge \mathrm{TU}_{\neq}(\bar{a}) \wedge \bigwedge_{\mathtt{i}=1}^{\ell} \forall qa(q \multimap_{\mathtt{i}} \bar{q} \wedge a \multimap_{\mathtt{i}} \bar{a} \to (\mathtt{CF}_{\mathrm{QT}}(q,x,a) \wedge a \prec t)))$

The classification of an object under a distributional property is defined in (Df20): at $t$, the object $x$ is classified under $\langle\hat{q},\hat{a}\rangle$ (where $\hat{a}$ is a spatial arrangement) when, following the *space* $\hat{a}$-path $\bar{a}$, the object $x$ *locally* follows the *quality* $\hat{q}$-path $\bar{q}$. According to (Df20), distributional properties hold in virtue of the spatial distribution of the qualities of the classified object.

**Df20** $\mathtt{sCF}(\langle\hat{q},\hat{a}\rangle,x,t) \triangleq \mathrm{SPR}(\langle\hat{q},\hat{a}\rangle) \wedge [\mathrm{SP}]\mathrm{PT}(\hat{a}) \wedge \exists\bar{q}\bar{a}(\mathtt{dCF}_{\mathrm{PT}}(\hat{q},\bar{q}) \wedge \mathtt{dCF}_{\mathrm{PT}}(\hat{s},\bar{a}) \wedge \mathrm{TU}_{\neq}(\bar{a}) \wedge \bigwedge_{\mathtt{i}=1}^{\ell} \forall qa(q \multimap_{\mathtt{i}} \bar{q} \wedge a \multimap_{\mathtt{i}} \bar{a} \to \mathtt{CF}_{\mathrm{QT}}(q,x,a,t)))$

Intuitively, the same object can be classified under a structural property by following different (cognitive) procedures, i.e., by following different paths. For instance, consider the distributional property $\langle\langle\mathrm{PWC},\langle\mathtt{color},\mathtt{color}\rangle\rangle,\langle\mathrm{PWC},\langle\mathtt{space},\mathtt{space}\rangle\rangle\rangle$, i.e., *having two connected colors at connected spaces*. Consider $\langle\mathtt{red},\mathtt{orange}\rangle$, $\langle\mathtt{orange},\mathtt{red}\rangle$, $\langle\mathtt{s}_1,\mathtt{s}_2\rangle$ and $\langle\mathtt{s}_2,\mathtt{s}_1\rangle$ (where these colors and these spaces are assumed to be connected). If an object is (at $t$) `red` at $\mathtt{s}_1$ and `orange` at $\mathtt{s}_2$ then, according to (Df20), we can follow $\langle\mathtt{red},\mathtt{orange}\rangle$ according to $\langle\mathtt{s}_1,\mathtt{s}_2\rangle$, or $\langle\mathtt{orange},\mathtt{red}\rangle$ according to $\langle\mathtt{s}_2,\mathtt{s}_1\rangle$. Thus, `sCF` concerns only how the object is at given spaces, not how it is (cognitively) explored. However, assume now the object is (at $t$) also `yellow` at space $\mathtt{s}_3$ (with $\mathtt{s}_2 \bowtie \mathtt{s}_3$). This object is classified by the above distributional property by following the paths $\langle\mathtt{red},\mathtt{orange}\rangle$ according to $\langle\mathtt{s}_1,\mathtt{s}_2\rangle$ or, alternatively, by following the paths $\langle\mathtt{orange},\mathtt{yellow}\rangle$ according to $\langle\mathtt{s}_2,\mathtt{s}_3\rangle$, i.e., the object exhibits the same structural property at different spaces. To avoid that, one could assure that the spatial location of the object (or its visible surface) is wholly covered by the spatial path. Analogous considerations hold for `tCF`.

## 7 Types

Gärdenfors's concepts are compounds of properties (belonging to different domains), i.e., they are regions in the multi-dimensional space composed by the domains. We extend this notion of concept (and conceptual space) by characterising our types in terms of both

[16] Note that $\mathtt{dCF}_{\mathrm{PT}}(\langle\sigma,\bar{c}\rangle,\bar{c})$ does not always hold.

[17] An internal relation R is a relation such that the truth-value of $\mathtt{R}(a,b)$ depends exclusively on similarity judgments along quality domains concerning the objects $a$ and $b$ only. See [13] for more details. Note that, our image schemas, through contexts and patterns, can apply to objects. This seems to contrast with the position in [11, 7], where images schemas, in particular the PATH schema, only apply to events or activities.

[18] Where $[\mathrm{D}_i]\mathrm{PT}(\langle\sigma,\bar{c}\rangle) \triangleq [\mathrm{D}_i]\mathrm{TU}(\bar{c})$.

[19] Clearly these properties are relevant also to characterize events (even though in this case the classification is not temporally qualified).

qualities (QT-instances) and structural properties (SPR-instances). The properties in the union of QT and SPR are called *features*. Notice that, even though distributional properties depend on how a given quality pattern is spatially arranged, they are still holistic, i.e., they apply to the whole objects. At the end of the section, we add $(i)$ *structured* properties (to be distinct from *structural* ones), i.e., following Armstrong, properties that hold in virtue of the way the classified object is mereologically structured into components of given types; and $(ii)$ *relational* properties, i.e., properties that hold in virtue of the way the classified object connects with the environment.

*Characterisation*—$\mathtt{CH}(x, y)$ reads as "the type $x$ is *characterised* by the feature (quality or structural property) $y$"—represents the link between types and features. Types have a multi-dimensional nature, they are characterised in terms of at least two, but usually several, domains (Ax21), where $[\mathrm{D}_i]\mathrm{SPR}(\langle\hat{q}, \hat{a}\rangle) \triangleq [\mathrm{D}_i]\mathrm{PT}(\hat{q})$.[20]

**Ax21** $\mathrm{TY}(x) \to \exists yz(\mathtt{CH}(x, y) \land \mathtt{CH}(x, z) \land$
$\bigvee_{i=1}^{\eta}((\mathrm{D}_i(y) \lor [\mathrm{D}_i]\mathrm{SPR}(y)) \land \neg\mathrm{D}_i(z) \land \neg[\mathrm{D}_i]\mathrm{SPR}(z)))$

The features of types are not ordered, types just *cluster* features (belonging to different domains). Indeed, these features are not necessarily atomic (see (Df2)), e.g., the Fiat500 type can be characterised by the red colour even though the colour-domain could contain more specific shades of red, like crimson, scarlet, etc.

Following the classical theory of types, one can assume that the features of a type express necessary and sufficient conditions to be classified by the type. In this classical perspective, the classification of an object by a type reduces to its classification by *all* its features. Structural qualities slightly complicate the scenario because depending on the kind of arrangement (temporal vs. spatial) one needs to consider $\mathtt{tCF}$ or $\mathtt{sCF}$. The definition of the classification of an object under a type ($\mathtt{CF}_{\mathrm{TY}}$) is given in (Df21).

**Df21** $\mathtt{CF}_{\mathrm{TY}}(x, y, t) \triangleq \mathrm{TY}(x) \land$
$\forall z(\mathtt{CH}(x, z) \to ((\mathrm{QT}(z) \to \mathtt{CF}_{\mathrm{QT}}(z, y, t)) \lor$
$([\mathrm{TM}]\mathrm{SPR}(z) \to \mathtt{tCF}(z, y, t)) \lor$
$([\mathrm{SP}]\mathrm{SPR}(z) \to \mathtt{sCF}(z, y, t)))$

There exist several possibilities to weaken the classical view on classification under types. For instance, one may introduce two characterisation-like relations to grasp the distinction between *necessary* and *optional* features of types, or introduce a 'weight' for each feature (as done for concepts in conceptual spaces). One may also distinguish the set of features sufficient to be classified by a type from the set of features sufficient to *not* be classified by a type.[21] Classification could also rely on the metrics of the domains (and on the metric of the overall space), i.e., one could see our notion of type as an extension of the one of *prototype* (see [24]) and capture classification (categorisation) on the basis of the distance between the qualities of the object and the features of the type.

Types can be organised by means of subsumption $\sqsubseteq_{\mathrm{TY}}$ (Df22). Given the definition of $\sqsubseteq_{\mathrm{SPR}}$ (Df18), $\sqsubseteq_{\mathrm{TY}}$ is grounded in the subsumption between qualities, i.e., following Gärdenfors, the way types are organised mainly depends on the organisation of qualities. Note that a type characterised by a (structural) quality can subsume only types characterised by (structural) qualities (in the same domain). It is trivial to prove that $\mathtt{CF}_{\mathrm{TY}}$ is closed under $\sqsubseteq_{\mathrm{TY}}$ (Th2), i.e., indirect classification under types is encapsulated into $\mathtt{CF}_{\mathrm{TY}}$.

**Df22** $x \sqsubseteq_{\mathrm{TY}} y \triangleq \mathrm{TY}(y) \land$
$\forall z(\mathtt{CH}(y, z) \to \exists w(\mathtt{CH}(x, w) \land (w \sqsubseteq z \lor w \sqsubseteq_{\mathrm{SPR}} z))$

**Th2** $\mathtt{CF}_{\mathrm{TY}}(x, y, t) \land x \sqsubseteq_{\mathrm{TY}} z \to \mathtt{CF}_{\mathrm{TY}}(z, y, t)$

As observed by Fiorini and colleagues [21], object recognition and categorisation are grounded not only in holistic properties but also in the identification of parts of the objects and the way these parts are structured. The number, the types, and the arrangement of parts are all essential aspects to found structural similarities between objects, an idea very close to the one of structured properties of Armstrong.

We start to sketch the formalisation of structured properties by extending the notion of pattern's context: patterns have the form $\langle\sigma, \bar{c}\rangle$ where $\bar{c}$ is now a DTU-tuple *or a tuple of types*. *Structured properties*, noted with TPR, have the same form of structural properties, i.e., $\langle\hat{p}, \hat{a}\rangle$, but now $\hat{p}$ is a pattern with a type-context.[22] We then consider a temporary parthood relation defined on objects: $\mathtt{PART}(x, y, t)$ stands for "the object $x$ is part of the object $y$ at time $t$" (see [12] for a FOL axiomatisation). Notice that, by excluding the possibility to have spatially co-localised objects, $\mathtt{PART}$ can be reduced to spatial inclusion.[23] The *local* classification by the type $p$ of the object $x$ as it is at space $s$ and time $t$ is the classification under $p$ of the part of $x$ that, at $t$, is exactly located at $s$ (Df23).[24]

**Df23** $\mathtt{CF}_{\mathrm{TY}}(p, x, s, t) \triangleq$
$\exists y(\mathtt{PART}(y, x, t) \land \mathrm{L}_{\mathrm{SP}}(y, s, t) \land \mathtt{CF}_{\mathrm{TY}}(p, y, t))$

The classification of an object under a structured property is defined in (Df24): at $t$, the object $x$ is classified under $\langle\hat{q}, \hat{a}\rangle$ (where $\hat{a}$ is a spatial arrangement and $\hat{q}$ a type-pattern) when, following the *space* $\hat{a}$-path $\bar{a}$, the object $x$ *mereologically* follows the *type* $\hat{q}$-path $\bar{q}$, i.e., its components selected by the spaces in $\bar{a}$ are instances of the correspondent types in $\bar{q}$.[25]

**Df24** $\mathtt{mCF}(\langle\hat{q}, \hat{a}\rangle, x, t) \triangleq \mathrm{TPR}(\langle\hat{q}, \hat{a}\rangle) \land [\mathrm{SP}]\mathrm{PT}(\hat{a}) \land$
$\exists\bar{q}\bar{a}(\mathtt{dCF}_{\mathrm{PT}}(\hat{q}, \bar{q}) \land \mathtt{dCF}_{\mathrm{PT}}(\hat{a}, \bar{a}) \land \mathrm{TU}_{\neq}(\bar{a}) \land$
$\bigwedge_{\mathtt{i}=1}^{\ell}\forall qa(q \multimap_{\mathtt{i}} \bar{q} \land a \multimap_{\mathtt{i}} \bar{a} \to \mathtt{CF}_{\mathrm{TY}}(q, x, a, t)))$

Our approach differs from the one of Fiorini and colleagues [21, 22] for two main reasons. First, in [21, 22] objects are always though as completely specified, i.e., as already noticed, an object is represented by a point in a conceptual space. By extending conceptual spaces with structural spaces an object comes already with all its parts (of given types) and the position of these parts with respect to the whole. Vice versa our framework can represent *partial information*, including mereological one, about objects. Second, and more importantly, [21, 22] focus on the representation of structural information in the framework of conceptual spaces where the similar-

[20] (Ax21) excludes the possibility to characterise a type in terms of both a $\mathrm{D}_i$-quality and a $[\mathrm{D}_i]\mathrm{SPR}$-property. This could be criticized. For instance, one could assume that an object with a given colour pattern could also have an holistic colour, i.e., the (conventional) colour of the whole object emerges from the colour pattern. This situation could be represented by providing a link between colour distributional properties and colour qualities, i.e., the classification under a colour quality would be inferred by the one under the corresponding strctural property. This is another interesting point that we do not address here.

[21] This approach can be represented by means of *rough sets* [19].

[22] Whether and how $\prec$ or $\bowtie$ can be defined on types is not taken into account in this paper. The only image schema that applies to tuples of types is the one that allows to built the paths.

[23] E.g., $\mathtt{PART}(x, y, t) \triangleq \exists sr(\mathrm{L}_{\mathrm{SP}}(x, s, t) \land \mathrm{L}_{\mathrm{SP}}(y, r, t) \land s \sqsubseteq r)$.

[24] $\mathtt{CF}_{\mathrm{TY}}(p, x, s, t)$ may also be introduced without reference to parts. The case of types characterised only in terms of qualities is a straightfoward generalisation of (Df20). The case of types characterised in terms of structural (and structured) properties requires to change (Df20) to assure the spaces considered in the tuple $\bar{a}$ to be included in $s$.

[25] Similarly to what noticed about (Df20) at the end of Sect. 6, (Df24) does not guarantee $x$ to be an assembly of only the components selected by $\bar{a}$, i.e., $x$ may have components not considered by the structured property $\langle\hat{q}, \hat{a}\rangle$.

ity relation is central. The construction mechanism behind structural spaces is purely mathematical, i.e., structural spaces are the mathematical product of the spaces that represent the parts and the ones that represent the whole-centered position of the parts. Our approach is more explicative, since it makes explicit the dependence of structural and structured properties from the instantiation of images schemas, namely, it offers a basis to explain structural similarities in terms of image schemas.

A *relational* property of an object $x$ can be seen as a structured property of an object (a system) $y$ that has $x$ among its components, i.e., it represents the way $x$ is linked to the other components of $y$. The *relational* classification of objects under structured properties is defined in (Df25).[26]

**Df25** $\mathtt{rCF}(\langle\!\langle \hat{q}, \hat{a} \rangle\!\rangle, x, t) \triangleq \mathrm{TPR}(\langle\!\langle \hat{q}, \hat{a} \rangle\!\rangle) \wedge [\mathrm{SP}]\mathrm{PT}(\hat{a}) \wedge$
$\exists y r \bar{q} \bar{a} (\mathtt{dCF}_{\mathrm{PT}}(\hat{q},\bar{q}) \wedge \mathtt{dCF}_{\mathrm{PT}}(\hat{a},\bar{a}) \wedge \mathrm{TU}_{\neq}(\bar{a}) \wedge$
$\mathrm{L}_{\mathrm{SP}}(x,r,t) \wedge \mathtt{PART}(x, y, t) \wedge r \multimap\!\!\diamond\, \bar{a} \wedge$
$\bigwedge_{\mathtt{i}=1}^{\ell} \forall qa (q \multimap_{\mathtt{i}} \bar{q} \wedge a \multimap_{\mathtt{i}} \bar{a} \rightarrow \mathtt{CF}_{\mathrm{TY}}(q, y, a, t)))$

(Df21) and (Df22) can then be extended to include structured and relational properties among the features that characterise types. These properties are fundamental to model assembled artefacts.

## 8 Conclusions

We presented a formal approach for the representation of compound types by modifying and extending Gärdenfors' theory of conceptual spaces. This is done by considering the notion of image schema and by generalising Gärdenfors' approach to patterns. In our framework, image schemas represent abstract (topological) structures, whereas patterns apply image schemas to specific quality-domains. Patterns are then used to specify either the historical properties of objects, namely constraints over their evolution in time, or objects' distributional or structured properties.

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