Laboratory for Applied Ontology

Institute of Cognitive Science and Technology
Italian National Research Council

Modeling in Knowledge Representation: the Parthood Relation

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Doctorate Course
2006-2007

Introduction

- Knowledge Representation (Artificial Intelligence)
  - Formal Ontology
  - Qualitative Reasoning
  - Temporal Reasoning
  - Spatial Reasoning
  - Natural Language Understanding
- Why the Parthood relation?
  - Philosophical, cognitive and linguistic relevance
  - Spatial and temporal reasoning based on vague information: impossibility to use exact coordinates, trajectories in terms of mathematical functions, and calculus
  - Reference to “extended” entities (e.g., temporal periods, spatial regions), possibly composed of parts of the same nature
  - No calculus, yet still a rigorous formal approach: logical theories

Outline of the course

- Mereology
- Time
- Mereotopology-1
- Exam topic discussion; Mereotopology-2
- Reasoning methods and complexity results
- Mereogeometry
- Mereogeometry

1-Mereology

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Formal Relation of Parthood

- Ontology
  - Domain of entities +
    Language (predicates + logic framework) +
    Properties (axioms)
- *Formal* Ontology
  - Formal framework, e.g., logic
  - Search for invariants across domains (Husserl)
- Basic structure
  - Unary predicates: implication or “is-a”
  - Entities: parthood relation
    - (my hand,my body), (today,this week), (this room,the university), (one student,the class), (mereology,formal ontology)…

A Bit of History

- Mereology
  - Lesniewski 1927-1931, *On the Foundations of Mathematics*
  - Greek *meros*
  - Alternative to Set Theory for escaping Russell's paradox
    - "the class of classes that are not members of themselves"
    - Expressed in a special logical language of its own "Ontology"
- Link with algebra: Tarski 1935
- The calculus of individuals, "nominalism"
  - Leonard & Goodman 1940
  - Expressed in first-order logic
  - No null individual
  - No abstract entities, no hierarchical distinction between individuals: a single relation of parthood
- Contemporary studies: Peter Simons (1986), Achille Varzi (1996)
  - All ontologies use a parthood relation, in the best cases fully specified with respect to Simons's work

Related structures: A math reminder

- Orders
- Lattices
- Boolean Algebras

Orders-1

- Comparison between entities
  - “is more/less … than”: “is bigger than”, “is smaller than”, “is later than”…
  - “is to the left of”, “is an ancestor of”, “is a divisor of”, “is part of”…
- Partial order, primitive $\leq$
  - First-order logic with identity
  - Reflexive (axiom): $\forall x\ x \leq x$
  - Transitive (axiom): $\forall xyz\ ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$
  - Antisymmetric (axiom): $\forall xy\ ((x \leq y \wedge y \leq x) \rightarrow x = y)$
  - Strict order (definition): $x < y \equiv_{df} x \leq y \wedge \neg y \leq x$
  - Inverse order (definition): $x \geq y \equiv_{df} y \leq x$ *no preferred direction*
- Strict partial order, primitive $<$
  - Transitive and asymmetric; irreflexive (theorem)
  - $x \leq y \equiv_{df} x < y \vee x = y$
- Choice of any of $\leq, \geq, <$, or $>$ as primitive

Orders-2

- Classical additional properties
  - Total order / linear order: $\forall xy\ (x \leq y \vee y \leq x)$
  - Discrete order:

    $\forall xy\ (x<y \rightarrow \exists z_1 z_2 (x<z_1 \wedge z_1 \leq y \wedge \forall t\ \neg(x<t \wedge t<z_1) \wedge x \leq z_2 \wedge z_2<y \wedge \forall t\ \neg(z_2<t \wedge t<y)))$

    on finite domains, all orders are discrete
  - Dense order: $\forall xy\ (x<y \rightarrow \exists z(x<z \wedge z<y))$
  - Bounded order: $\exists x_1 x_2\ \forall y\ (x_1 \leq y \wedge y \leq x_2)$

    Bounded to the left, bounded to the right
  - Unbounded order: $\forall x\ \exists y_1 y_2\ (y_1<x \wedge x<y_2)$
- Examples: $\langle N,\leq\rangle$; $\langle Z,\leq\rangle$; $\langle Q,\leq\rangle$; $\langle R,\leq\rangle$

Orders-3

- Many orders on many domains!
  - weights, heights, numbers, instants (precedence), preferences…
  - ➢ these are not parthood!
- Specificities of Parthood?
  - Surely not a linear order
  - Dense, discrete, bounded, unbounded: all possible options
  - So?
- Well-known order, much related to parthood: set inclusion
- Yet, no need to refer to the membership relation
- ➢ Lattices and algebras

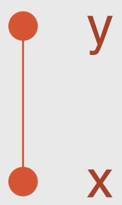
Lattices

- Set equipped with a partial order s.t. any two entities have an infimum and a supremum
  - $\forall xy\ \exists z(z \le x \wedge z \le y \wedge \forall t\ ((t \le x \wedge t \le y) \rightarrow t \le z)$ z is noted $x \wedge y$ *meet*
  - $\forall xy\ \exists z(x \le z \wedge y \le z \wedge \forall t\ ((x \le t \wedge y \le t) \rightarrow z \le t)$ z is noted $x \vee y$ *join*
  - Semilattices: join semilattices, meet semilattices
- Set equipped with two operators, meet ($\wedge$) and join ($\vee$) s.t.
  - $\forall xy\ (x \wedge y = y \wedge x \ \wedge\ x \vee y = y \vee x)$ *commutativity*
  - $\forall xyz\ ((x \wedge y) \wedge z = x \wedge (y \wedge z) \ \wedge\ (x \vee y) \vee z = x \vee (y \vee z))$ *associativity*
  - $\forall xy\ (x \wedge (x \vee y) = x \ \wedge\ x \vee (x \wedge y) = x)$ *absorption*
  - Theorem: $\forall x\ (x \wedge x = x \ \wedge\ x \vee x = x)$ *idempotence*
  - $x \le y \equiv_{df} x = x \wedge y$, or equivalently, $x \le y \equiv_{df} y = x \vee y$
- Examples: $\langle 2^D, \cap, \cup \rangle$; $\langle \text{Prop}, \wedge, \vee \rangle$; $\langle \{\bot, T\}, \wedge, \vee \rangle$; $\langle N^*, \text{gcd}, \text{lcm} \rangle$; $\langle N^*, \min, \max \rangle$
- Equivalence of the two definitions: $\langle N, \le \rangle$ same lattice as $\langle N, \min, \max \rangle$

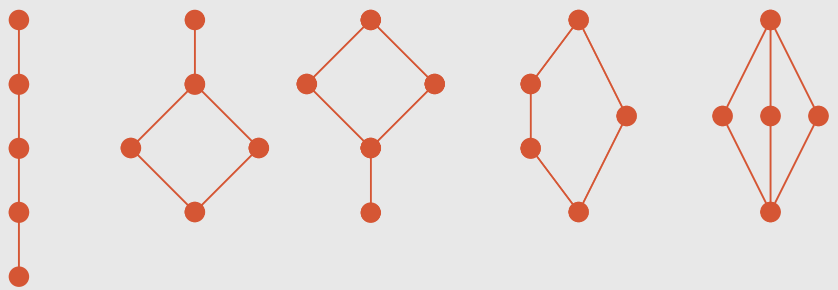
Hasse diagrams

- Graph on finite domains
- Convention: all vertical or oblique arcs are implicitly oriented from bottom to top, strict order

x<y



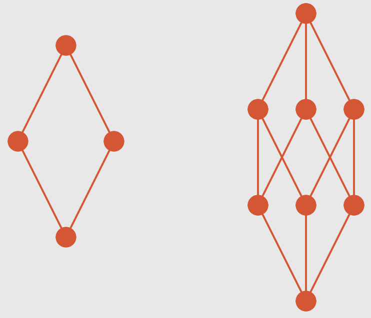
- All lattices with 5 elements



Lattices and Boolean Algebras

- Distributive lattice
  - $\forall xyz\ (x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \wedge \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z))$
- Complemented lattice
  - $\exists x_1 x_2\ \forall y\ (y \wedge x_1 = x_1\ \wedge\ y \vee x_2 = x_2)$ x_1 noted 0 or $\perp$; x_2 noted 1 or T
  - Complement operator, noted ' or - s.t. $\forall x\ (x \wedge x' = 0\ \wedge\ x \vee x' = 1)$

- Boolean algebra = complemented distributed lattice
- Examples: $\langle$Prop, $\wedge$, $\vee$, $\neg$, $\perp$, T$\rangle$; $\langle\{\perp$, T$\}$, $\wedge$, $\vee$, $\neg$, $\perp$, T$\rangle$;
  $\langle 2^D, \cap, \cup, ^-, \varnothing, D\rangle$



Boolean Algebras-2

- De Morgan's rules are theorems
  - $(x \wedge y)' = x' \vee y'$
  - $(x \vee y)' = x' \wedge y'$
- Atoms
  - $At(x) \equiv_{df} \neg x=0 \wedge \forall y \, (y \leq x \rightarrow (y=0 \vee y=x))$
- Atomic algebras
  - $\forall x \, \exists y \, (At(y) \wedge y \leq x)$
- Complete lattices
  - Any subset has a supremum and an infimum
  - ➢ Finite lattices are complete

From algebras to Mereology

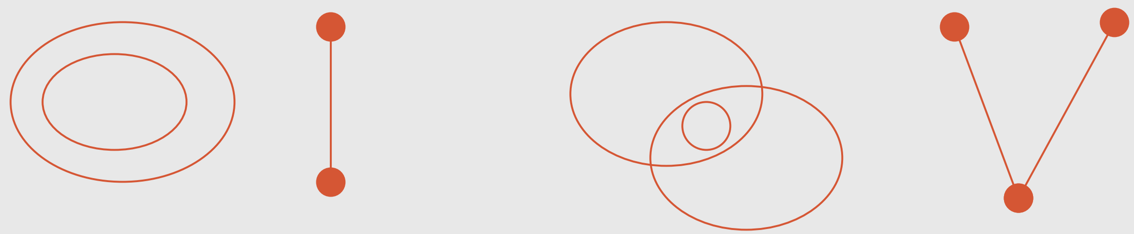
- No “null” entity

- Not necessarily existence of the universe
- Not necessarily existence of the join and the meet

Yet… properties related to those of Boolean algebras

Basic Mereology: M

- P, partial order *part*
  - (M1) $\forall x\, P(x,x)$
  - (M2) $\forall xyz\, ((P(x,y) \wedge P(y,z)) \rightarrow P(x,z))$
  - (M3) $\forall xy\, ((P(x,y) \wedge P(y,x)) \rightarrow x=y)$
- Definitions
  - $PP(x,y) \equiv_{df} P(x,y) \wedge \neg P(y,x)$ *proper part*
  - $O(x,y) \equiv_{df} \exists z\, (P(z,x) \wedge P(z,y))$ *overlap*
  - $PO(x,y) \equiv_{df} O(x,y) \wedge \neg P(x,y) \wedge \neg P(y,x))$ *proper overlap*

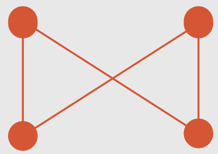


Extensional Mereology

- Supplementation

(M4) $\forall xy\,(PP(x,y) \rightarrow \exists z\,(P(z,y) \wedge \neg O(z,x)))$

Weak supplementation



(M5) $\forall xy\,(\neg P(y,x) \rightarrow \exists z\,(P(z,y) \wedge \neg O(z,x)))$

Strong supplementation

- Extensionality

  (E1) $\forall xy\,((\exists z\,PP(z,x) \wedge \forall z\,(PP(z,x) \leftrightarrow PP(z,y))) \rightarrow x=y)$

  (E2) $\forall xy\,(\forall z\,(O(z,x) \leftrightarrow O(z,y)) \rightarrow x=y)$

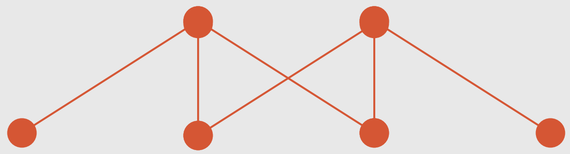
- Theorems
  - M+(M5) $\vdash$ (M4); M+(M5) $\vdash$ (E1); M+(M5) $\vdash$ (E2);
  - M+(M4) $\nvdash$ (E1); M+ (E1) $\nvdash$ (E2);
  - M+(E2) $\vdash$ (M4)

Closure Mereology - 1

- Product

  (M6) $\forall xy\,(O(x,y) \rightarrow \exists z\,\forall t\,(P(t,z) \leftrightarrow (P(t,x) \wedge P(t,y))))$

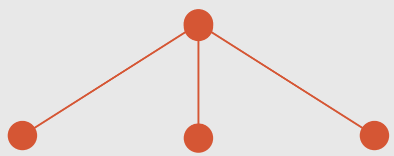
  - z is *the* product of x and y, noted x•y



- Sum

  (M7) $\forall xy\,\exists z\,\forall t\,(O(t,z) \leftrightarrow (O(t,x) \vee O(t,y)))$

  - (E2) entails the unicity of z
  - z is the sum of x and y, noted x+y



Closure Mereology - 2

- Difference

  (M8) $\forall xy\ (\exists z\ (P(z,x) \wedge \neg O(z,y)) \rightarrow \exists z\ \forall t\ (P(t,z) \leftrightarrow (P(t,x) \wedge \neg O(t,y))))$

  - z is the difference of x and y, noted x-y
- Complement
  - Existence of the universe, noted U

  (M9) $\exists x\ \forall y\ P(y,x)$

  - Definition of the complement: $\sim x = U - x$ (exists for all $x \neq U$)

Classical/General Extensional Mereology

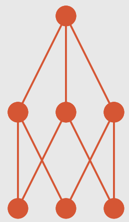
- General fusion

  (M6') $\exists x\, \phi(x) \rightarrow \exists z\, \forall y\, (O(y,z) \leftrightarrow \exists x\, (\phi(x) \wedge O(y,x)))$

  - Axiom schema, useful for infinite domains
  - unicity guaranteed by (E2), z is noted $\sigma x\, \phi(x)$
- Russell's description operator ι often used
  - $\sigma x\, \phi(x) = \iota z\, \forall y\, (O(y,z) \leftrightarrow \exists x\, (\phi(x) \wedge O(y,x)))$
- Sum, product and complement as fusions
  - $x+y = \sigma z\, (P(z,x) \vee P(z,y))$
  - $x \cdot y = \sigma z\, (P(z,x) \wedge P(z,y))$
  - $\sim x = \sigma z\, (\neg O(z,x))$
- Universe
  - $U = \sigma x\, P(x,x)$
- No null element! No general fusion of nothing: $\neg \exists z\, (z = \sigma x\, \neg P(x,x))$

Mereology and Algebra

- General Extensional Mereology characterizes complete Boolean algebras with the null element removed [Tarski 1935], that is, complete distributed complemented lattices without bottom.



Atomicity

- Atoms
  - $At(x) \equiv_{df} \forall y\ (P(y,x) \rightarrow y=x)$
- Atomicity

  (AT1) $\forall x\ \exists y\ (At(y) \wedge P(y,x))$
- Atomic essentialism

  (AT2) $\forall xy\ (\forall z\ (At(z) \rightarrow (P(z,x) \rightarrow P(z,y))) \rightarrow P(x,y))$
- Theorems

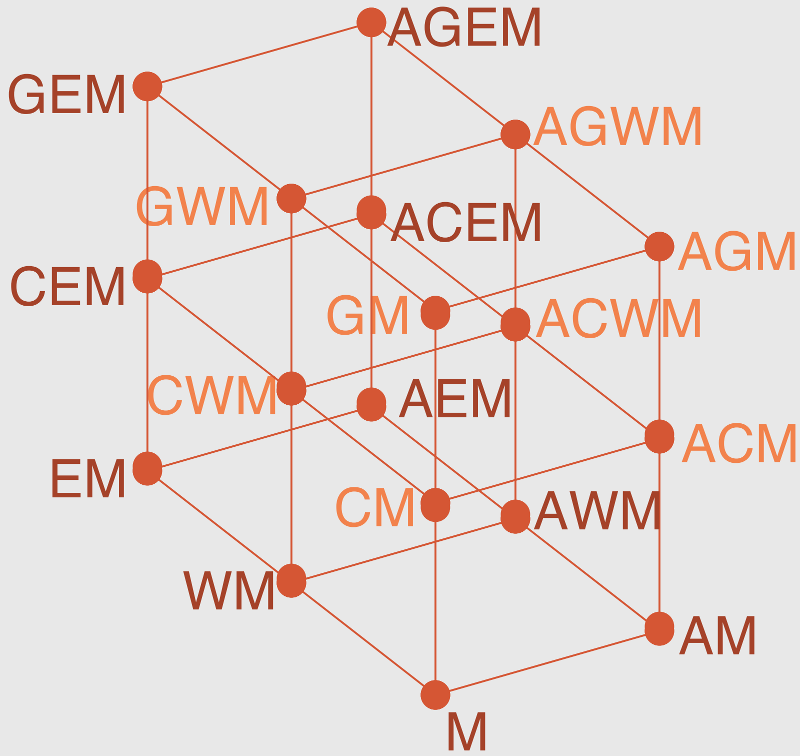
  M+(M5)+(AT1) $\vdash$ (AT2)

  M+(M5)+(AT2) $\vdash$ (AT1)

  M+(AT2) $\vdash$ (M5)

  M+(E1)+(AT1) $\vdash$ (E2)

Mereologies



Questioning Classical Extensional Mereology

- Some mereotopologies reject even weak supplementation (Lectures 3&4)
- Extensionality
  - Loosing or acquiring parts: identity across time
  - Identity between my body and the collection of my organs
- Closure: sum of my nose and Caesar's toe
  Fusion: even stranger scattered infinite sums
  - First move: mereotopology to identify "wholes"
- Transitivity?
  - My hand is part of me, I'm part of the ICT School College, but my hand is not part of the College
  - The handle is part of the door, the door is part of the house. Is the handle part of the house?

Distinguishing various Part-Whole relations

- Linguistic and psychological evidence
  - Lyons 1977, Cruse 1986, Winston et al. 1987…
- Part-whole relations and *meronomies*
- A set of relations
  - Member / collection
    - This cow / the herd, John / the orchestra
  - Sub-collection / collection
    - Benelux / EU (but not USA / NATO)
  - Component-Integral Whole
    - The handle / the door, the engine / my car
  - Portion-Whole
    - A piece of cake
  - Substance-Whole
    - Some sugar / this cake
  - Piece-Whole
    - The left half of this table