Modeling in Knowledge Representation: The Parthood Relation

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Doctorate Course
2006-2007

4-Mereotopology (Part 2)

Doctorate Course
Modeling in Knowledge Representation: The Parthood Relation
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Why Mereotopology ?

Formalizing common-sense knowledge
- proved to be much harder than formalized expert knowledge
- is based on a common-sense ontology
- that includes objects of every day live
- rather than sets.

Mathematics (Topology)
- uses set theory to represent real world problems
- provides sophisticated tools for expert reasoning.

Topology: A Reminder

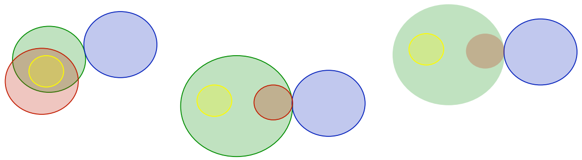
Definition
- A *topology* on a carrier set D is (given by) a set $\mathcal{T} \subseteq 2^D$ having the following properties
  - D, $\emptyset \in \mathcal{T}$
  - $\forall S \subseteq \mathcal{T} [\bigcup S \in \mathcal{T}]$
  - $\forall X, Y \in \mathcal{T} [X \cap Y \in \mathcal{T}]$
- The elements of $\mathcal{T}$ are called *open sets.*

Simple Examples
- {D, Ø} (trivial topology)
- 2^D (discrete topology)
- $\mathcal{T}_d = \{\bigcup M \mid M \subseteq \mathcal{B}_d\}$ (metric based topology)

Spatial Structure: Mereotopological Calculi

Basic idea

- (extended) regions are basic entities in the spatial ontology
- topological structure is crucial for spatial structure (→ qualitative)
- points and boundaries are abstractions from configurations of regions



Mereotopologies

A large selection of proposals exist

- Whitehead (1929), Clarke (1981), Randell & Cohn (1989), Egenhofer (1991), Randell Cui & Cohn (RCC, 1992), Vieu (1993), Asher & Vieu (A&V, 1995), Roeper (1997), Eschenbach (CRC, 1999), Borgo, Guarino, Masolo (BGM, 1996)

- Is there a common core to the proposals?
- Which approaches can be combined?
- How to choose between 'the proposals' for an application?

How to proceed?

Select terminologies

- Use notation that is neutral regarding the theories
- Identify the common terminological kernel
- Distinguish terms whose definitions differ
  - Mereological terminology (def. based on Part-of <)
  - Topological terminology (def. based on contact C)
  - Mereotopological terminology (def. based on < and C)

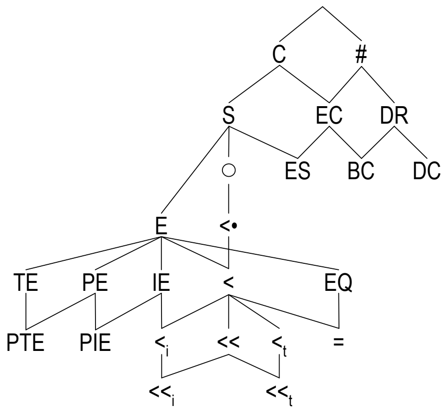
Identify a list of axioms

- such that every approach can be identified with a subset

Analyse the interrelation between the axioms

Select Terminologies

(Selection of) Binary Relations in Mereotopology



Identify a List of Axioms

Study interactions

Basic Assumptions Collected

The axioms used up to now are

M: $\forall x\, y\, z\, [z < x \wedge x < y \Rightarrow z < y]$

M: $\forall x\, y\, [x < y \wedge y < x \Rightarrow x = y]$

M: $\forall x\, [x < x]$

T: $\forall x\, [C(x, x)]$

T: $\forall x\, y\, [C(x, y) \Rightarrow C(y, x)]$

M+T: $\forall x\, y\, [x < y \Rightarrow E(x, y)]$

Types of Axioms

- ✓ Principles of extensionality, supplementation (individuation)
- ✓ Interaction between mereological and topological terminology
- ✓ Topological and mereological universes
- ✓ Closed regions
- Existence of open regions; divisibility
- A first map
- Atoms and connectedness of space
- Three types of complements and connectedness
- Mereological and topological completeness

Extensionality and Interaction of Mereological and Topological Terms

Extensionality and the interaction between mereology and topology

ET: Extensional Topology

$\forall x\, y\, [EQ(x, y) \Rightarrow x = y]$

ET $\models \forall x\, y\, [x \neq y \Leftrightarrow \exists z\, [C(x, z) \Leftrightarrow \neg C(y, z)]]$

EM: Strong supplementation (Extensional Mereology)

$\forall x\, y\, [x <\!\bullet\, y \Rightarrow x < y]$

$\forall x\, y\, [\forall z\, [z \circ x \Rightarrow z \circ y] \Rightarrow x < y]$

EM $\models \forall x\, y\, [x \neq y \Leftrightarrow \exists z\, [z \circ x \Leftrightarrow \neg (z \circ y)]]$

M+ST: Strong Mereotopological Interaction

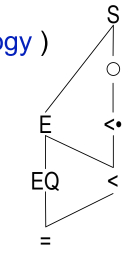
$\forall x\, y\, [E(x, y) \Rightarrow x < y]$

$\forall x\, y\, [\forall z\, [C(z, x) \Rightarrow C(z, y)] \Rightarrow x < y]$

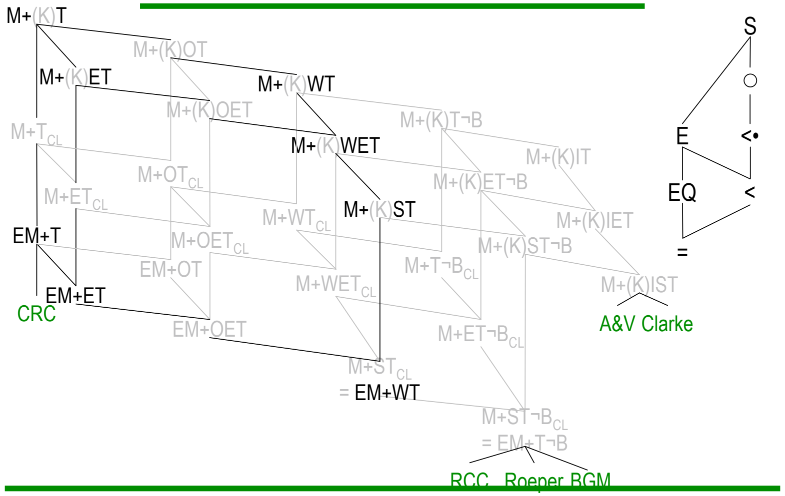
M+WT: Weak Mereotopological Interaction

$\forall x\, y\, [E(x, y) \Rightarrow x <\!\bullet\, y]$

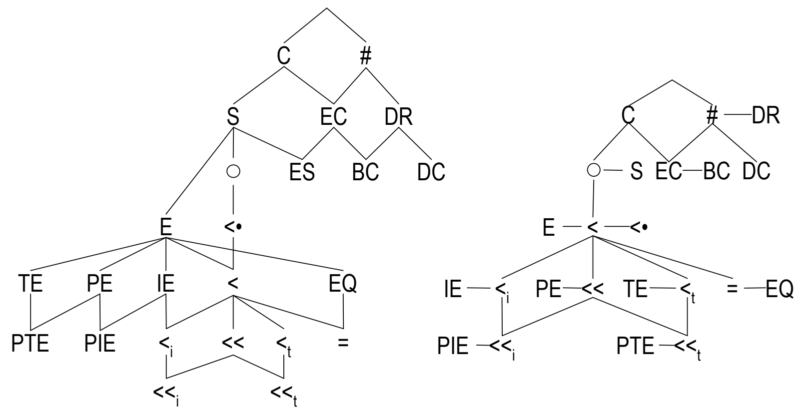
M+WT $\models \forall x\, y\, [S(x, y) \Leftrightarrow x \circ y]$



Principles of extensionality do not contradict each other



Terminological Simplifications in EM+WT



Mereologically and Topologically Universal Regions

Universal Regions (2)

Different versions for Mereology and Topology

m-univ(x) $\Leftrightarrow_{def}$ ∀y [y ○ x]

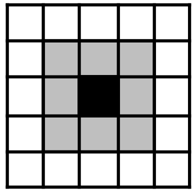
t-univ(x) $\Leftrightarrow_{def}$ ∀y [C(y, x)]

M+T |= ∀x [m-univ(x) ⇒ t-univ(x)]

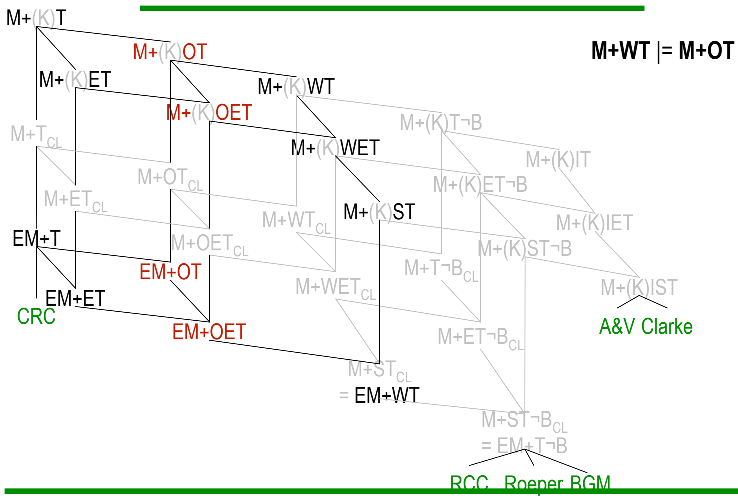
M+OT: Topologically universal regions are mereologically universal

∀x [t-univ(x) ⇒ m-univ(x)]

- The grey area is topologically universal but not mereologically universal.



(Second Part of) A first map

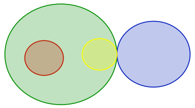


Closed and Open Regions

Closed and Open Regions

Point-Set Topology

- Closed sets include all their boundaries.
  - Missing connections of closed regions derive from missing parts.

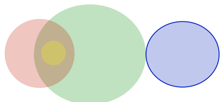


CL(x) $\Leftrightarrow_{def}$ $\forall y$ [y <• $x \Rightarrow$ E(y, x)]

M+T |= $\forall x$ [CL(x) $\Leftrightarrow$ $\forall y$ [$\exists z$ [C(z, y) $\wedge$ ¬C(z, x)] $\Rightarrow$ $\exists w$ [$w < y \wedge$ ¬(w ○ x)]]]

- Open sets do not include their boundaries.
  - All connections derive from sharing parts.
  - There are no external connections to open regions.

OP(x) $\Leftrightarrow_{def}$ ¬$\exists y$ [EC(y, x)]



Closed Regions and Closures

A region is *closed* iff it encloses all regions it covers.

CL(x) $\Leftrightarrow_{def}$ $\forall y$ [y <• $x \Rightarrow$ E(y, x)]

M+T$_{CL}$: All regions are closed

$\forall x$ [CL(x)]

A *closure* of region y is connected to exactly those regions that are connected to a region y covers.

cl(x; z) $\Leftrightarrow_{def}$ $\forall u$ [C(u, x) $\Leftrightarrow$ $\exists y$ [y <• z $\wedge$ C(u, y)]]

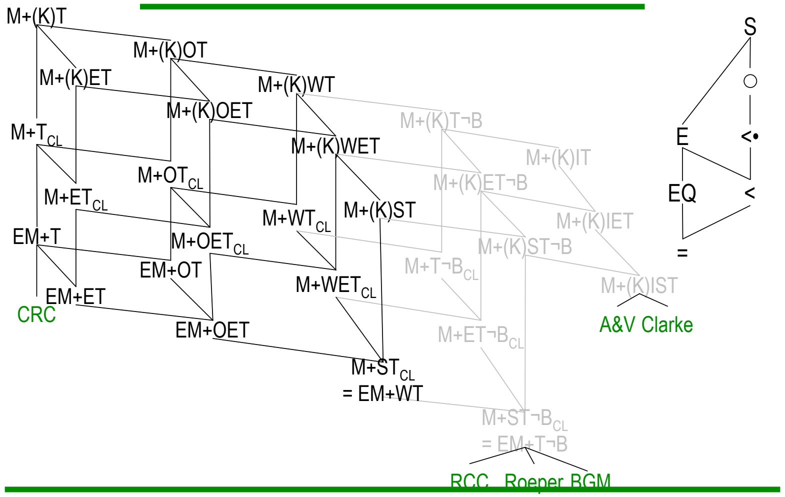
M+T |= $\forall x\, y$ [cl(x; z) $\Rightarrow$ $\forall y$ [y <• $z \Rightarrow$ E(y, x)]

M+T |= $\forall x$ [CL(x) $\Leftrightarrow$ cl(x; x)]

M+KT: All regions have a closure

$\forall y\, \exists x$ [cl(x; y)]

Principles of extensionality do not contradict each other
Mereological extensionality stronger than closedness of regions



Open Regions, inner parts and interiors

A region is *open*, iff it has no external contacts.

OP(x) $\Leftrightarrow_{def}$ ¬$\exists y$ [EC(y, x)]

M+T |= $\forall x$ [OP(x) $\Leftrightarrow$ $\forall z$ [C(z, x) $\Rightarrow$ z ○ x]]

M+T |= $\forall x$ [OP(x) $\Leftrightarrow$ $x <_i x$]

$x <_i y \Leftrightarrow_{def} x < y \wedge \forall z$ [C(z, x) $\Rightarrow$ z ○ y]

A theory where all regions are open is *discrete*.

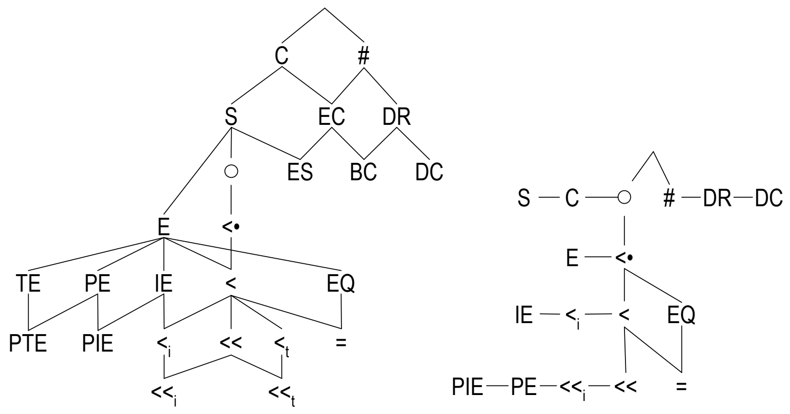
DISCRETE $\Leftrightarrow_{def}$ $\forall x$ [OP(x)]

An *interior* of a region y is a region connected to exactly those regions that overlap y.

int(x; y) $\Leftrightarrow_{def}$ $\forall z$ [C(z, x) $\Leftrightarrow$ z ○ y]

M+T |= $\forall x$ [OP(x) $\Leftrightarrow$ int(x; x)]

Terminological Simplifications in Discrete Spaces



Open Regions, Inner Parts and Interiors

A region is *open*, iff it has no external contacts.

$OP(x) \Leftrightarrow_{def} \neg \exists y \, [EC(y, x)]$

An *interior* of a region *y* is a region connected to exactly those regions that overlap *y*.

$int(x; y) \Leftrightarrow_{def} \forall z \, [C(z, x) \Leftrightarrow z \circ y]$

M+IT: All regions have an interior

$\forall y \, \exists x \, [int(x; y)]$

M+T¬B: All regions have an inner part

$\forall x \, \exists y \, [y <_i x]$

$x <_i y \Leftrightarrow_{def} x < y \wedge \forall z \, [C(z, x) \Rightarrow z \circ y]$

Open Regions, Inner Parts and Interiors

A region is *open*, iff it has no external contacts.

$OP(x) \Leftrightarrow_{def} \neg \exists y \, [EC(y, x)]$

An *interior* of a region *y* is a region connected to exactly those regions that overlap *y*.

$int(x; y) \Leftrightarrow_{def} \forall z \, [C(z, x) \Leftrightarrow z \circ y]$

$\models \forall x \, y \, z \, [int(x; z) \Rightarrow (int(y; z) \Leftrightarrow EQ(x, y))]$

$\models \forall x \, y \, [EQ(x, y) \Rightarrow \forall z \, [int(x; z) \Leftrightarrow int(y; z)]]$

$\models \forall x \, y \, [x <\!\bullet\!> y \Rightarrow \forall z \, [int(z; x) \Leftrightarrow int(z; y)]]$

$\models \forall x \, y \, z \, [int(z; x) \Rightarrow (int(z, y) \Leftrightarrow x <\!\bullet\!> y)]$

Open Regions, Inner Parts and Interiors

A region is *open*, iff it has no external contacts.

$OP(x) \Leftrightarrow_{def} \neg \exists y \, [EC(y, x)]$

An *interior* of a region *y* is a region connected to exactly those regions that overlap *y*.

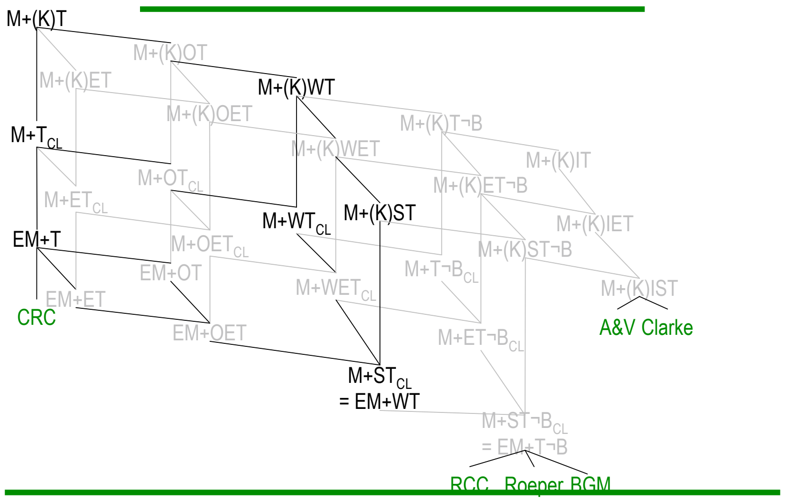
$int(x; y) \Leftrightarrow_{def} \forall z \, [C(z, x) \Leftrightarrow z \circ y]$

M+T $\models \forall x \, [OP(x) \Leftrightarrow int(x; x)]$

M+T $\models \forall x \, y \, [int(x; y) \Rightarrow IE(x, y)]$

M+T $\models \forall x \, y \, [int(x; y) \Rightarrow x <\!\bullet \, y]$

Selection from (Third Part of) A first map



Theories with inner parts and interiors

M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x <\!\bullet\, y]$

M+ST: $\forall x\, y\, [E(x, y) \Rightarrow x < y]$

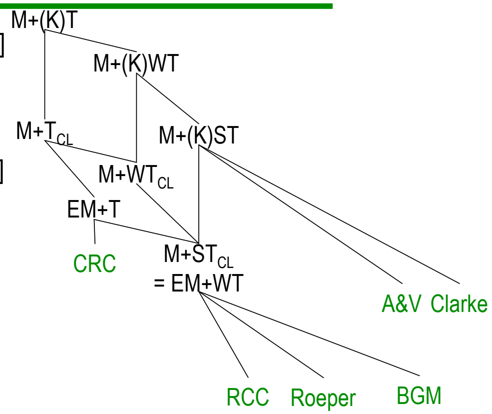
M+KT: $\forall y\, \exists x\, [cl(x; y)]$

M+T$_{CL}$: $\forall x\, y\, [y <\!\bullet\, x \Rightarrow E(y, x)]$

EM: $\forall x\, y\, [x <\!\bullet\, y \Rightarrow x < y]$

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

M+IT: $\forall y\, \exists x\, [int(x; y)]$



Theories with inner parts and interiors

M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x <\!\bullet\, y]$

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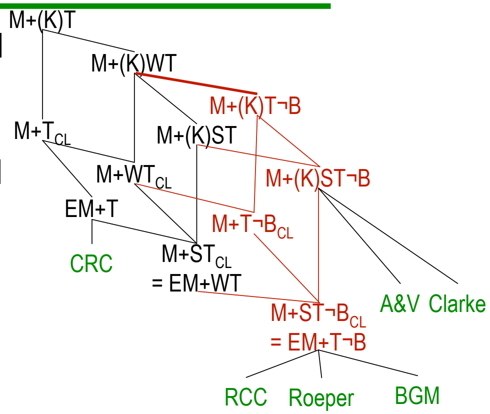
M+T¬B: $\forall x\, \exists y\, [y <_i x]$

M+IT: $\forall y\, \exists x\, [int(x; y)]$

M+T¬B |= M+WT



Theories with inner parts and interiors

M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x <\!\bullet\, y]$

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M+KT: $\forall y\, \exists x\, [cl(x; y)]$

M+T$_{CL}$: $\forall x\, y\, [y <\!\bullet\, x \Rightarrow E(y, x)]$

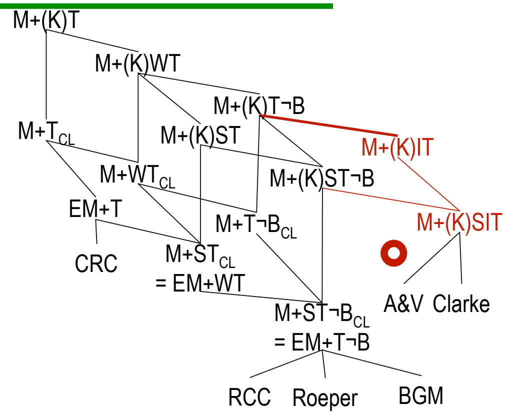
EM: $\forall x\, y\, [x <\!\bullet\, y \Rightarrow x < y]$

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

M+IT: $\forall y\, \exists x\, [int(x; y)]$

M+IT |= M+T¬B

M+IT$_{CL}$ |= DISCRETE

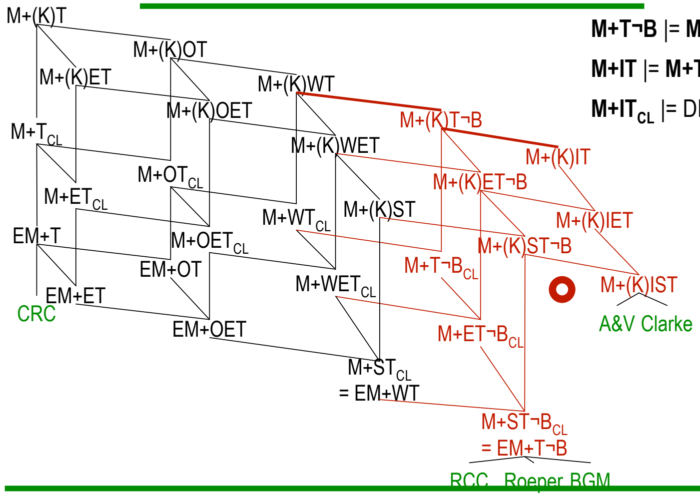


A first map

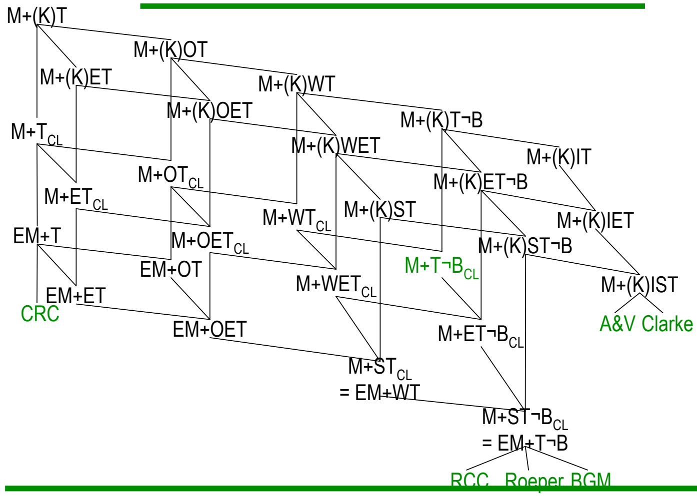
$\mathbf{M+T\neg B} \models \mathbf{M+WT}$

$\mathbf{M+IT} \models \mathbf{M+T\neg B}$

$\mathbf{M+IT_{CL}} \models \mathrm{DISCRETE}$



The first Map with some Landmarks



Some Remarks on M+T¬B

All regions have interior parts.

- In this theory, the mereotopological operators behave similar to their topological counterparts on regular sets.

All interiors are open. All closures are closed.

$\mathbf{M+T\neg B} \models \forall x\, y\, [\mathrm{int}(x; y) \Rightarrow \mathrm{OP}(x)]$

$\mathbf{M+T\neg B} \models \forall x\, y\, [\mathrm{cl}(x; y) \Rightarrow \mathrm{CL}(x)]$

int and cl do not add region-parts

$\mathbf{M+T\neg B} \models \forall x\, y\, [\mathrm{int}(x; y) \Rightarrow x <\!\bullet\!> y]$

$\mathbf{M+T\neg B} \models \forall x\, y\, [\mathrm{cl}(x; y) \Rightarrow x <\!\bullet\!> y]$

Some Remarks on M+T¬B

All regions have interior parts.

- In this theory, the mereotopological operators behave similar to their topological counterparts on regular sets.

double cl and int

$\mathbf{M+T\neg B} \models \forall x\, y\, z\, [\mathrm{cl}(y; z) \Rightarrow (\mathrm{cl}(x; y) \Leftrightarrow \mathrm{EQ}(x; y))]$

$\mathbf{M+T\neg B} \models \forall x\, y\, z\, [\mathrm{int}(y; z) \Rightarrow (\mathrm{int}(x; y) \Leftrightarrow \mathrm{EQ}(x; y))]$

regularity

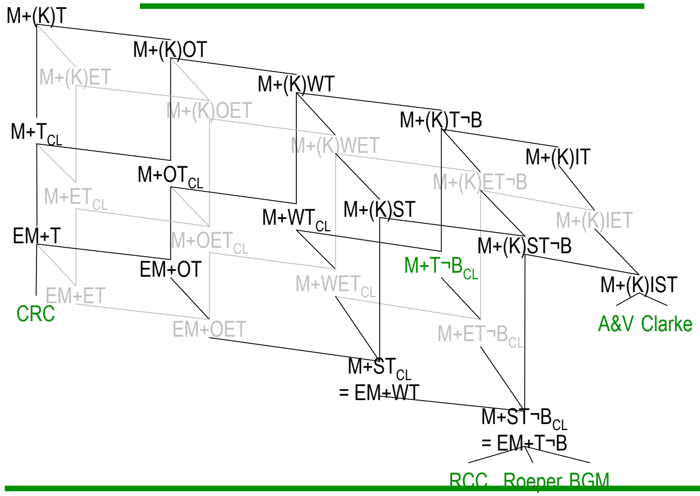
$\mathbf{M+T\neg B} \models \forall x\, y\, z\, [\mathrm{cl}(z; y) \Rightarrow (\mathrm{int}(x; y) \Leftrightarrow \mathrm{int}(x; z))]$

$\mathbf{M+T\neg B} \models \forall x\, y\, z\, [\mathrm{int}(z; y) \Rightarrow (\mathrm{cl}(x; y) \Leftrightarrow \mathrm{cl}(x; z))]$

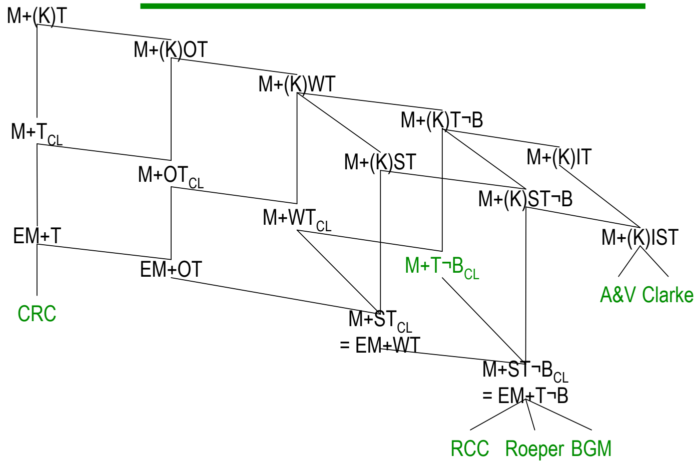
interiors are the fusions of the inner parts

$\mathbf{M+T\neg B} \models \forall x\, y\, [\mathrm{int}(x; y) \Leftrightarrow \forall z\, [C(x, z) \Leftrightarrow \exists w\, [C(w, z) \wedge w <_i y]]]$

An Interesting Part of the First Map



A Simplified Map



Connected Space

Connectedness of Space and Atomic Regions

A region that is both open and closed is *isolated* (from the rest of space)

$\text{ISO}(x) \Leftrightarrow_{def} \text{OP}(x) \wedge \text{CL}(x)$

M+T $\models \forall x\, [\text{m-univ}(x) \Rightarrow \text{ISO}(x)]$

M+T$_{CON}$: Only mereologically universal regions are isolated

$\forall x\, [\text{ISO}(x) \Rightarrow \text{m-univ}(x)]$

A region without proper parts is an *atom*

$\text{At}(x) \Leftrightarrow_{def} \neg \exists y\, [y \ll x]$

MA: Every region has an atomic part

$\forall y\, \exists x\, [\text{At}(x) \wedge x < y]$

Atoms in M+T¬B$_{CL}$

A region without proper parts is an *atom*

At(x) $\Leftrightarrow_{def}$ $\neg\exists y\,[y \ll x]$

M+T¬B $\models \forall x\,[\text{At}(x) \Rightarrow \text{OP}(x)]$

M+T¬B$_{CL}$ $\models \forall x\,[\text{At}(x) \Rightarrow \text{ISO}(x)]$

MA+T¬B$_{CL}$ $\models \forall y\,[\exists x\,[x < y \wedge \text{ISO}(x)]]$

M+T¬B$_{CON\ CL}$ $\models \forall x\,[\text{At}(x) \Rightarrow \text{m-univ}(x)]$

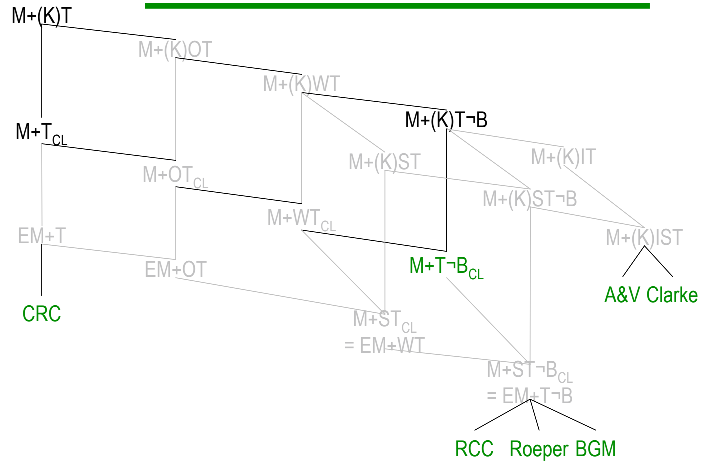
M+T¬B$_{CON\ CL}$ $\models \forall x\,[\text{At}(x) \Rightarrow \forall y\,[x < y]]$

M+T¬B$_{CON\ CL}$, $\exists x\,[\text{At}(x)] \models \forall y\,[\text{m-univ}(y)]$

M+T¬B$_{CON\ CL}$, $\exists x\,[\text{At}(x)] \models$ DISCRETE

MA+T¬B$_{CON\ CL}$ $\models$ DISCRETE

A Critical Region in the Simplified Map



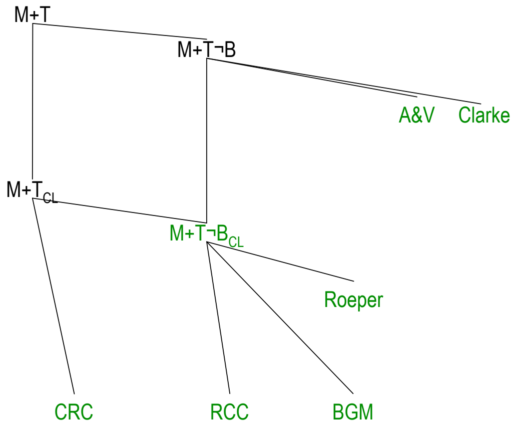
Connectedness of Space and Atomic Regions

M+T¬B: $\forall x\,\exists y\,[y <_i x]$

M+T$_{CL}$:
$\forall x\,y\,[y <\bullet\, x \Rightarrow E(y, x)]$

M+T$_{CON}$:
$\forall x\,[\text{ISO}(x) \Rightarrow \text{m-univ}(x)]$

MA: $\forall y\,\exists x\,[\text{At}(x) \wedge x < y]$



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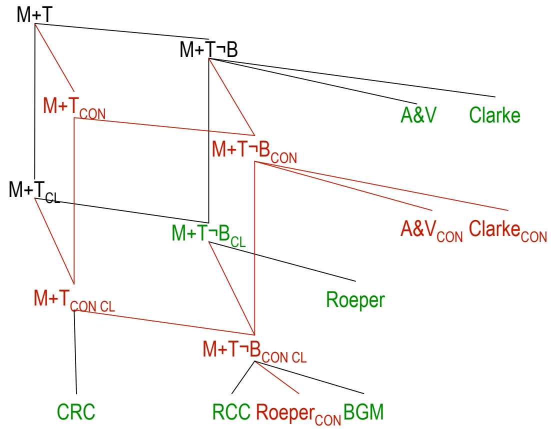

Connectedness of Space and Atomic Regions

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

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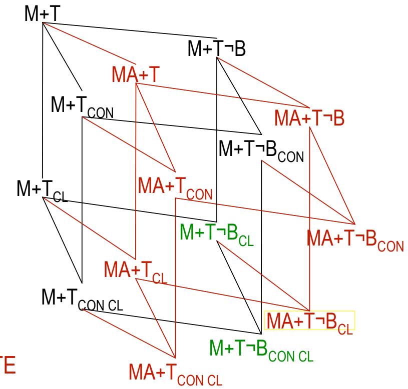
M+T$_{CON}$:
$\forall x\, [ISO(x) \Rightarrow m\text{-}univ(x)]$

MA: $\forall y\, \exists x\, [At(x) \wedge x < y]$

M+T¬B$_{CON\ CL}$, $\exists x\, [At(x)]$
|= DISCRETE

MA+T¬B$_{CON\ CL}$ |= DISCRETE



Connected Spaces and Atomic Regions

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

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M+T¬B$_{CON\ CL}$, $\exists x\, [At(x)]$
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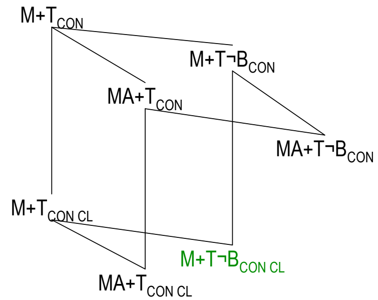

Connected Spaces and Atomic Regions

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

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MA: $\forall y\, \exists x\, [At(x) \wedge x < y]$

M+T¬B$_{CON\ CL}$, $\exists x\, [At(x)]$
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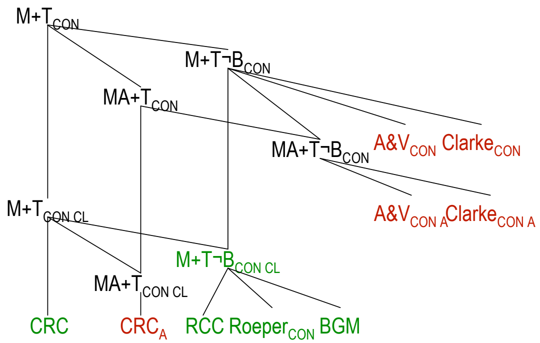
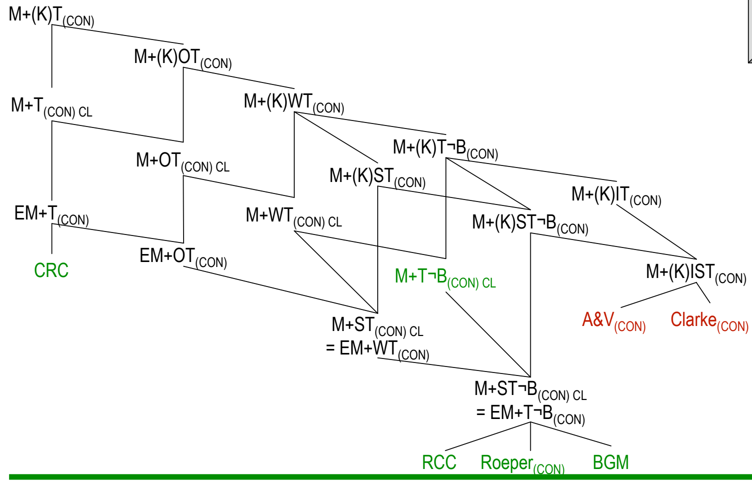
MA+T¬B$_{CON\ CL}$ |= DISCRETE



A Simplified Map (of Connected Space)



Complements

Three Types of Complements

A *mereological complement* of region y overlaps exactly those regions that are not covered by y

$$\text{m-compl}(x; y) \Leftrightarrow_{def} \forall w\, [x \circ w \Leftrightarrow \neg(w <\!\bullet\, y)]$$

A *topological complement* x of a region y is connected to exactly those regions that are not enclosed by y

$$\text{t-compl}(x; y) \Leftrightarrow_{def} \forall z\, [C(z, x) \Leftrightarrow \neg E(z, y)]$$

A *closed complement* of region y is connected to those regions that are not proper inner parts of y and overlaps those regions that are not part of y.

$$\text{c-compl}(x; y) \Leftrightarrow_{def} \forall z\, [C(z, x) \Leftrightarrow \neg(z <<_i y)] \wedge \forall z\, [z \circ x \Leftrightarrow \neg(z < y)]$$

Mereological Complements

A *mereological complement* of region y overlaps exactly those regions that are not covered by y

$$\text{m-compl}(x; y) \Leftrightarrow_{def} \forall w\, [x \circ w \Leftrightarrow \neg(w <\!\bullet\, y)]$$

$\mathbf{M} \models \forall x\, y\, [\text{m-compl}(x; y) \Rightarrow \neg(x \circ y)]$

$\mathbf{M} \models \forall x\, y\, z\, [\text{m-compl}(x; y) \Rightarrow (\neg(x \circ z) \Rightarrow y \circ z)]$

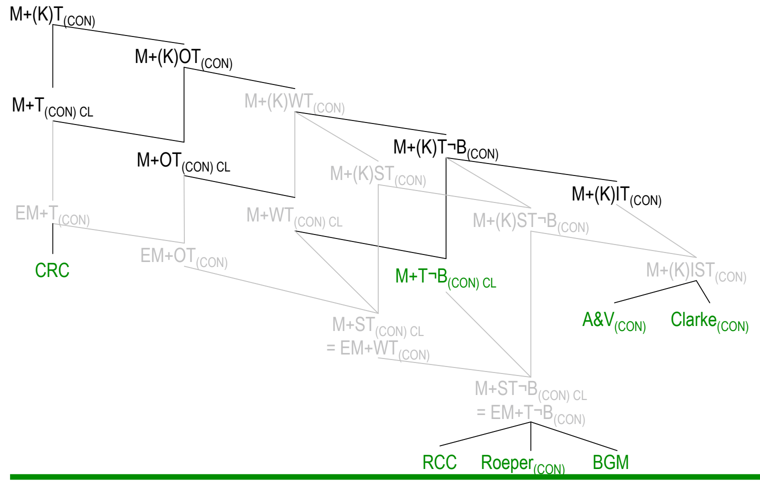
$\mathbf{M} \models \forall x\, y\, [\text{m-compl}(x; y) \Leftrightarrow \text{m-compl}(y; x)]$

$\mathbf{M} \models \forall x\, y\, [\text{m-compl}(x; y) \Rightarrow \neg\text{m-univ}(x) \wedge \neg\text{m-univ}(y)]$

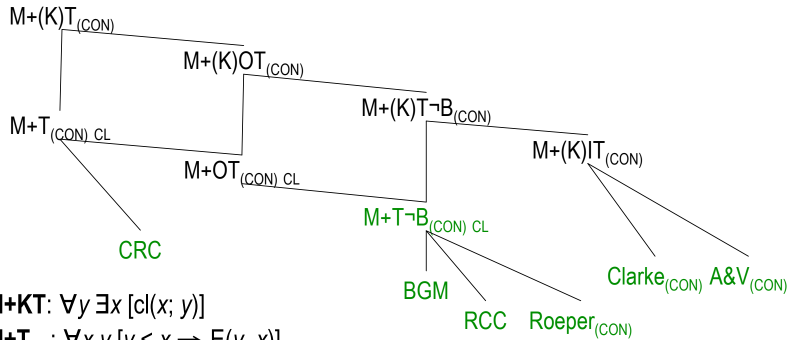
XM+T: Existence of mereological complements

$$\forall y\, [\neg\, \text{m-univ}(y) \Rightarrow \exists x\, [\text{m-compl}(x; y)]]$$

A Critical Region in the Simplified Map



Mereological Complements



M+KT: $\forall y\, \exists x\, [\mathrm{cl}(x; y)]$

M+T$_{CL}$: $\forall x\, y\, [y < x \Rightarrow \mathrm{E}(y, x)]$

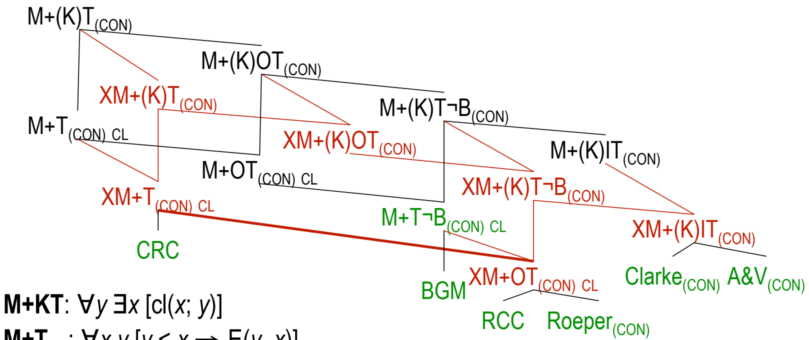
M+OT: $\forall x\, [\text{t-univ}(x) \Rightarrow \text{m-univ}(x)]$

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

M+IT: $\forall y\, \exists x\, [\mathrm{int}(x; y)]$

XM: $\forall y\, [\neg \text{m-univ}(y) \Rightarrow \exists x\, [\text{m-compl}(x; y)]]$

Mereological Complements



M+KT: $\forall y\, \exists x\, [\mathrm{cl}(x; y)]$

M+T$_{CL}$: $\forall x\, y\, [y < x \Rightarrow \mathrm{E}(y, x)]$

M+OT: $\forall x\, [\text{t-univ}(x) \Rightarrow \text{m-univ}(x)]$

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

M+IT: $\forall y\, \exists x\, [\mathrm{int}(x; y)]$

XM: $\forall y\, [\neg \text{m-univ}(y) \Rightarrow \exists x\, [\text{m-compl}(x; y)]]$

XM+OT$_{CL}$ $\models$ **M+T¬B**

Topological Complements

A *topological complement* *x* of a region *y* is connected to exactly those regions that are not enclosed by *y*

$\text{t-compl}(x; y) \Leftrightarrow_{def} \forall z\, [\mathrm{C}(z, x) \Leftrightarrow \neg \mathrm{E}(z, y)]$

T $\models \forall x\, y\, [\text{t-compl}(x; y) \Rightarrow \neg \mathrm{C}(x, y)]$

M+T $\models \forall x\, y\, [\text{t-compl}(x; y) \Rightarrow \neg(y \circ x)]$

T $\models \forall x\, y\, z\, [\text{t-compl}(x; y) \Rightarrow (\neg \mathrm{C}(x, z) \Rightarrow \mathrm{C}(y, z))]$

T $\models \forall x\, y\, [\text{t-compl}(x; y) \Leftrightarrow \text{t-compl}(y; x)]$

T $\models \forall x\, y\, [\text{t-compl}(x; y) \Rightarrow \neg \text{t-univ}(x) \wedge \neg \text{t-univ}(x)]$

M+T $\models \forall x\, y\, [\text{t-compl}(x; y) \wedge \mathrm{OP}(y) \Rightarrow \mathrm{CL}(x)]$

Some Remarks on M+T¬B

All regions have interior parts.

M+T¬B $\models \forall x\, y\, [\text{t-compl}(x; y) \Rightarrow \text{m-compl}(x; y)]$

- In this theory, the mereotopological operators are dual regarding topological complement.

M+T¬B $\models \forall x\, y\, [\text{t-compl}(x; y) \Rightarrow (\mathrm{CL}(y) \Leftrightarrow \mathrm{OP}(x))]$

M+T¬B $\models \forall x\, y\, u\, z\, [\text{t-compl}(x; u) \wedge \text{t-compl}(z; y) \Rightarrow (\mathrm{cl}(x; y) \Leftrightarrow \mathrm{int}(u; z))]$

Topological Complements

A *mereological complement* of region *y* overlaps exactly those regions that are not covered by *y*

$m\text{-compl}(x; y) \Leftrightarrow_{def} \forall w\,[x \circ w \Leftrightarrow \neg(w <\bullet y)]$

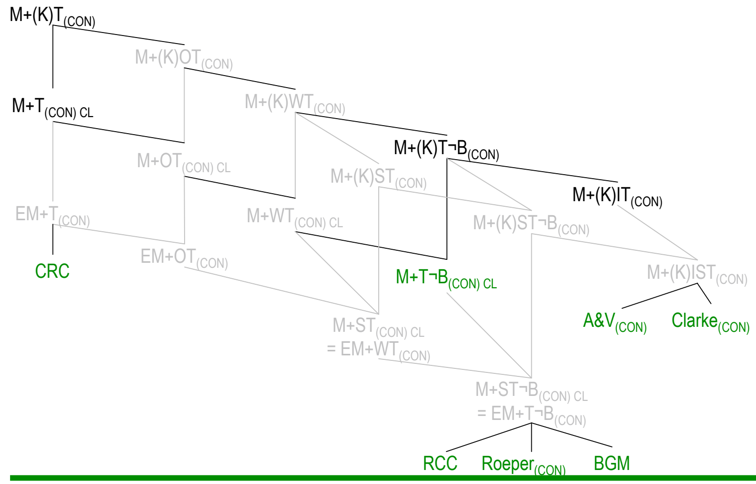
A *topological complement* *x* of a region *y* is connected to exactly those regions that are not enclosed by *y*

$t\text{-compl}(x; y) \Leftrightarrow_{def} \forall z\,[C(z, x) \Leftrightarrow \neg E(z, y)]$

M+XT: Existence of topological complements

$\forall y\,[\neg t\text{-univ}(y) \Rightarrow \exists x\,[t\text{-compl}(x; y)]]$

A Critical Region in the Simplified Map



Topological Complements

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

M+IT: $\forall y\, \exists x\, [int(x; y)]$

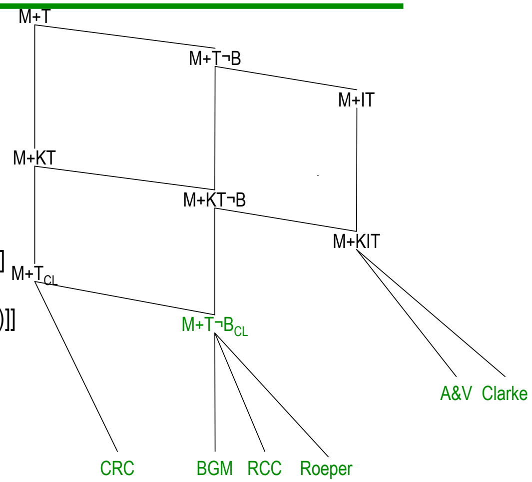
M+KT: $\forall y\, \exists x\, [cl(x; y)]$

M+T$_{CL}$: $\forall x\, y\, [y <\bullet x \Rightarrow E(y, x)]$

XT: $\forall y\, [\neg t\text{-univ}(y) \Rightarrow \exists x\, [t\text{-compl}(x; y)]]$

XM: $\forall y\, [\neg m\text{-univ}(y) \Rightarrow \exists x\, [m\text{-compl}(x; y)]]$

M+IT$_{CL}$ |= DISCRETE



Topological Complements

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

M+IT: $\forall y\, \exists x\, [int(x; y)]$

M+KT: $\forall y\, \exists x\, [cl(x; y)]$

M+T$_{CL}$: $\forall x\, y\, [y <\bullet x \Rightarrow E(y, x)]$

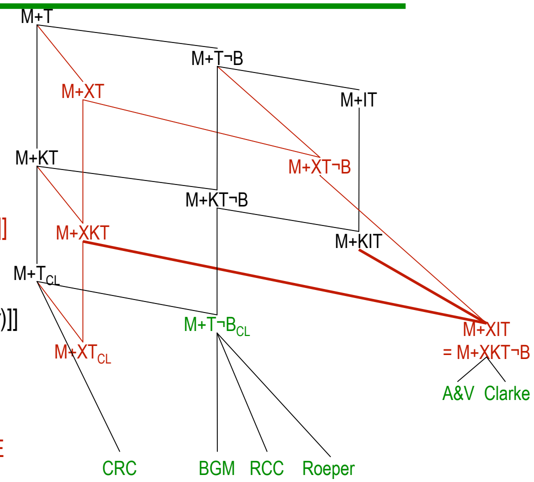
XT: $\forall y\, [\neg t\text{-univ}(y) \Rightarrow \exists x\, [t\text{-compl}(x; y)]]$

XM: $\forall y\, [\neg m\text{-univ}(y) \Rightarrow \exists x\, [m\text{-compl}(x; y)]]$

M+XIT |= **M+KT**

M+XKT¬B |= **M+IT**

M+XT¬B$_{CL}$ |= DISCRETE



Topological and Mereological Complements

M+T¬B: ∀x ∃y [$y <_i x$]

M+IT: ∀y ∃x [int(x; y)]

M+KT: ∀y ∃x [cl(x; y)]

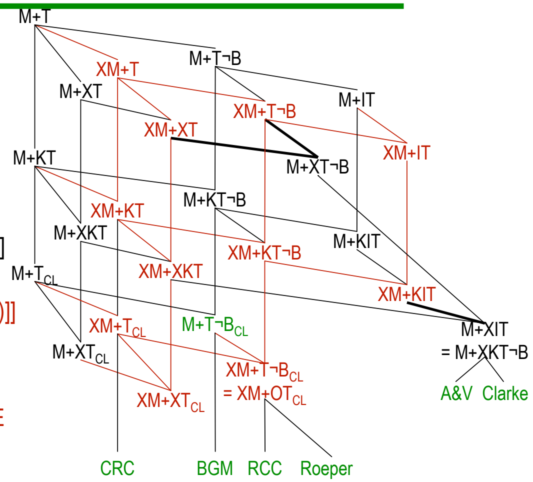
M+T$_{CL}$: ∀x y [y <• x
⇒ E(y, x)]

XT: ∀y [¬t-univ(y)
⇒ ∃x [t-compl(x; y)]]

XM: ∀y [¬m-univ(y)
⇒ ∃x [m-compl(x; y)]]

M+XT¬B |= **XM**

XM+XOT$_{CL}$ |= DISCRETE



Closed Complements

A *closed complement* of region y is connected to those regions that are not proper inner parts of y and overlaps those regions that are not part of y.

c-compl(x; y) $\Leftrightarrow_{def}$ ∀z [C(z, x) ⇔ ¬($z <<_i y$)] ∧ ∀z [z ○ x ⇔ ¬($z < y$)]

M+T |= ∀x y [c-compl(x; y) ⇒ CL(x) ∧ CL(y)]

M+T |= ∀x y [c-compl(x; y) ⇒ EC(x, y)]

M+T |= ∀x y [c-compl(x; y) ⇒ m-compl(x; y)]

M+ST |= ∀x y [c-compl(x; y) ⇒ c-compl(y; x)]

EM+T$_{CON}$ |= ∀x y [m-compl(x; y) ⇔ c-compl(x; y)]

Closed Complements

A *mereological complement* of region y overlaps exactly those regions that are not covered by y

m-compl(x; y) $\Leftrightarrow_{def}$ ∀w [x ○ w ⇔ ¬(w <• y)]

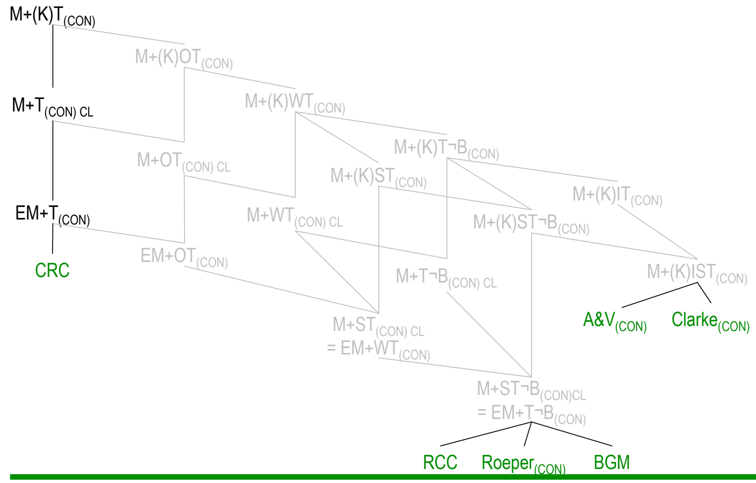
A *closed complement* of region y is connected to those regions that are not proper inner parts of y and overlaps those regions that are not part of y.

c-compl(x; y) $\Leftrightarrow_{def}$ ∀z [C(z, x) ⇔ ¬($z <<_i y$)] ∧ ∀z [z ○ x ⇔ ¬($z < y$)]

X'M+T: Existence of closed complements

∀y [¬m-univ(y) ⇒ ∃x [c-compl(x; y)]]

A Selection from the Simplified Map (of Connected Space)



Complements and Connectedness

M+KT: $\forall y\, \exists x\, [\mathrm{cl}(x; y)]$

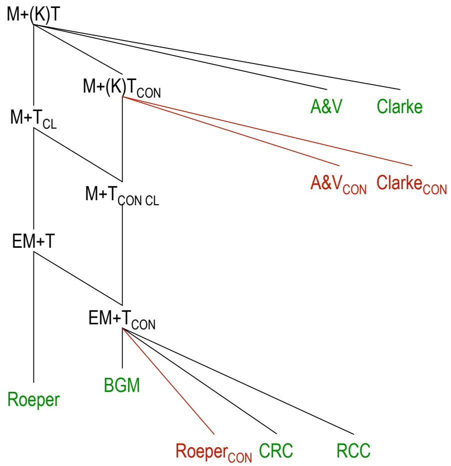
$M+T_{CL}$: $\forall x\, y\, [y \lessdot x \Rightarrow E(y, x)]$

EM: $\forall x\, y\, [x \lessdot y \Rightarrow x < y]$

$M+T_{CON}$: $\forall x\, [\mathrm{ISO}(x) \Rightarrow \text{m-univ}(x)]$

XM: $\forall y\, [\neg\, \text{m-univ}(y)$
$\Rightarrow \exists x\, [\text{m-compl}(x; y)]]$

X'M: $\forall y\, [\neg\, \text{m-univ}(y)$
$\Rightarrow \exists x\, [\text{c-compl}(x; y)]]$



Complements and Connectedness

M+KT: $\forall y\, \exists x\, [\mathrm{cl}(x; y)]$

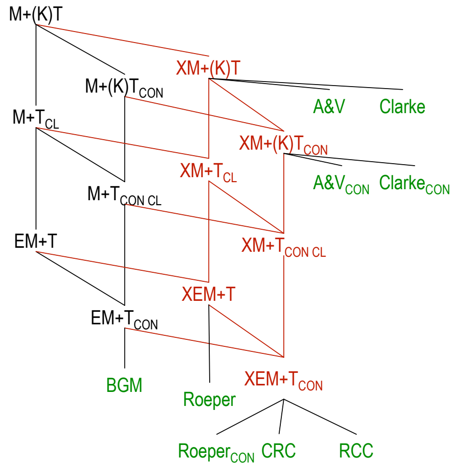
$M+T_{CL}$: $\forall x\, y\, [y \lessdot x \Rightarrow E(y, x)]$

EM: $\forall x\, y\, [x \lessdot y \Rightarrow x < y]$

$M+T_{CON}$: $\forall x\, [\mathrm{ISO}(x) \Rightarrow \text{m-univ}(x)]$

XM: $\forall y\, [\neg\, \text{m-univ}(y)$
$\Rightarrow \exists x\, [\text{m-compl}(x; y)]]$

X'M: $\forall y\, [\neg\, \text{m-univ}(y)$
$\Rightarrow \exists x\, [\text{c-compl}(x; y)]]$



Complements and Connectedness

M+KT: $\forall y\, \exists x\, [\mathrm{cl}(x; y)]$

$M+T_{CL}$: $\forall x\, y\, [y \lessdot x \Rightarrow E(y, x)]$

EM: $\forall x\, y\, [x \lessdot y \Rightarrow x < y]$

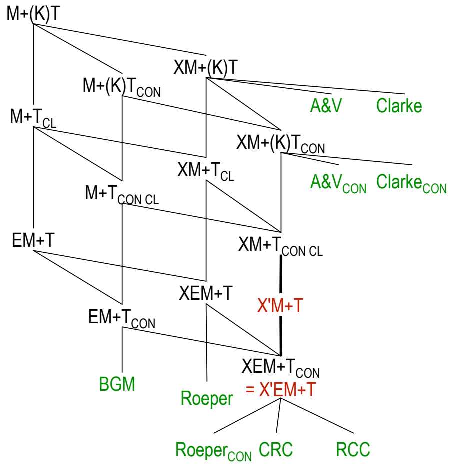
$M+T_{CON}$: $\forall x\, [\mathrm{ISO}(x) \Rightarrow \text{m-univ}(x)]$

XM: $\forall y\, [\neg\, \text{m-univ}(y)$
$\Rightarrow \exists x\, [\text{m-compl}(x; y)]]$

X'M: $\forall y\, [\neg\, \text{m-univ}(y)$
$\Rightarrow \exists x\, [\text{c-compl}(x; y)]]$

$X'M+T \models XM+T_{CON\ CL}$

$XEM+T_{CON} \models X'M+T$



All Complements at one Glance

M+T¬B: $\forall x\, \exists y\, [y <_i x]$

M+IT: $\forall y\, \exists x\, [\mathrm{int}(x; y)]$

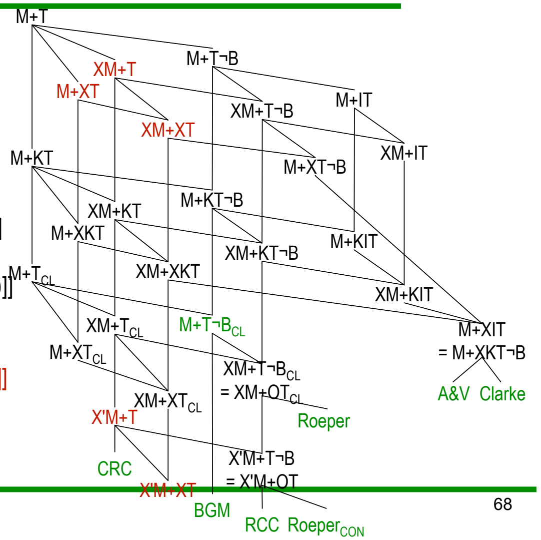
M+KT: $\forall y\, \exists x\, [\mathrm{cl}(x; y)]$

$M+T_{CL}$: $\forall x\, y\, [y \lessdot x$
$\Rightarrow E(y, x)]$

XT: $\forall y\, [\neg\, \text{t-univ}(y)$
$\Rightarrow \exists x\, [\text{t-compl}(x; y)]]$

XM: $\forall y\, [\neg\, \text{m-univ}(y)$
$\Rightarrow \exists x\, [\text{m-compl}(x; y)]]$

X'M: $\forall y\, [\neg\, \text{m-univ}(y)$
$\Rightarrow \exists x\, [\text{c-compl}(x; y)]]$



Mereological and Topological Completeness

Mereological Sums, Intersections and Differences

Overlap-based definitions

(D) $\text{m-fus}(x; y, z) \Leftrightarrow_{def} \forall w\, [w \bigcirc x \Leftrightarrow (w \bigcirc y \vee w \bigcirc z)]$

(D) $\text{m-isct}(x; y, z) \Leftrightarrow_{def} \forall w\, [w \bigcirc x \Leftrightarrow \exists v\, [w \bigcirc v \wedge v < y \wedge v < z]]$

(D) $\text{m-diff}(x; y, z) \Leftrightarrow_{def} \forall w\, [w \bigcirc x \Leftrightarrow \exists v\, [w \bigcirc v \wedge v < y \wedge \neg(v \bigcirc z)]]$

(D) $x\ \sigma y\, [\Phi] \Leftrightarrow_{def} \forall z\, [z \bigcirc x \Leftrightarrow \exists y\, [\Phi \wedge z \bigcirc y]]$

CM: Binary sums and intersections exist, if possible

$\forall y\, z\, \exists x\, [\text{m-fus}(x; y, z)]$

$\forall y\, z\, [y \bigcirc z \Rightarrow \exists x\, [\text{m-isct}(x; y, z)]]$

DM: Differences exist, if possible

$\forall y\, z\, [\neg(y <\!\bullet\, z) \Rightarrow \exists x\, [\text{m-diff}(x; y, z)]]$

GM: Arbitrary sums exist, if possible

$\exists y\, [\Phi] \Rightarrow \exists x\, \forall z\, [x \bigcirc z \Leftrightarrow \exists y\, [\Phi \wedge y \bigcirc z]]$

Mereological Completeness Conditions

CM: $\forall y\, z\, \exists x\, [\text{m-fus}(x; y, z)]$

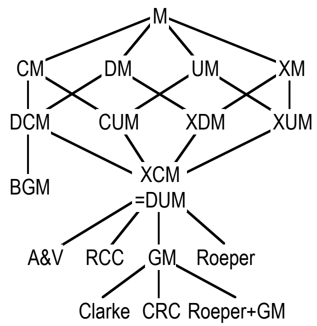
CM: $\forall y\, z\, [y \bigcirc z \Rightarrow \exists x\, [\text{m-isct}(x; y, z)]]$

DM: $\forall y\, z\, [\neg(y <\!\bullet\, z) \Rightarrow \exists x\, [\text{m-diff}(x; y, z)]]$

GM: $\exists y\, [\Phi] \Rightarrow \exists x\, \forall z\, [x \bigcirc z \Leftrightarrow \exists y\, [\Phi \wedge y \bigcirc z]]$

XM: $\forall y\, [\neg \text{m-univ}(y) \Rightarrow \exists x\, [\text{m-compl}(x; y)]]$

UM+T: $\exists x\, [\text{m-univ}(x)]$



XCM |= DUM

DUM |= XCM

GM |= XDCUM

Topological Sums and Intersections

Connection-based definitions

(D) $\text{t-fus}(x; y, z) \Leftrightarrow_{def} \forall w\, [C(w, x) \Leftrightarrow (C(w, y) \vee C(w, z))]$

(D) $\text{t-isct}(x; y, z) \Leftrightarrow_{def} \forall w\, [C(w, x) \Leftrightarrow \exists v\, [C(w, v) \wedge E(v, y) \wedge E(v, z)]]$

CT: Binary sums and intersections exist, if possible

$\forall y\, z\, \exists x\, [\text{t-fus}(x; y, z)]$

$\forall y\, z\, [S(y, z) \Rightarrow \exists x\, [\text{t-isct}(x; y, z)]]$

GT: Arbitrary sums exist, if possible

$\exists y\, [\Phi] \Rightarrow \exists x\, \forall z\, [C(x, z) \Leftrightarrow \exists y\, [\Phi \wedge C(y, z)]]$

Interaction between mereological and topological sum and intersection principles

From topological to mereological completeness

M+WT |= ∀*x y z* [t-isct(*x*; *y*, *z*) ⇒ m-isct(*x*; *y*, *z*)]

M+T¬B |= ∀*x y z* [t-fus(*x*; *y*, *z*) ⇒ m-fus(*x*; *y*, *z*)]

M+CT¬B |= CM

From mereological to topological completeness

M+WT$_{CL}$ |= ∀*x y z* [m-isct(*x*; *y*, *z*) ⇔ t-isct(*x*; *y*, *z*)]

M+FT: Missing connections mean missing parts, generalized

∀x y z [∃u [C(u, x) ∧ ¬(C(u, y) ∨ C(u, z))] ⇒ ∃w [w < x ∧ ¬(w ○ y ∨ w ○ z)]]

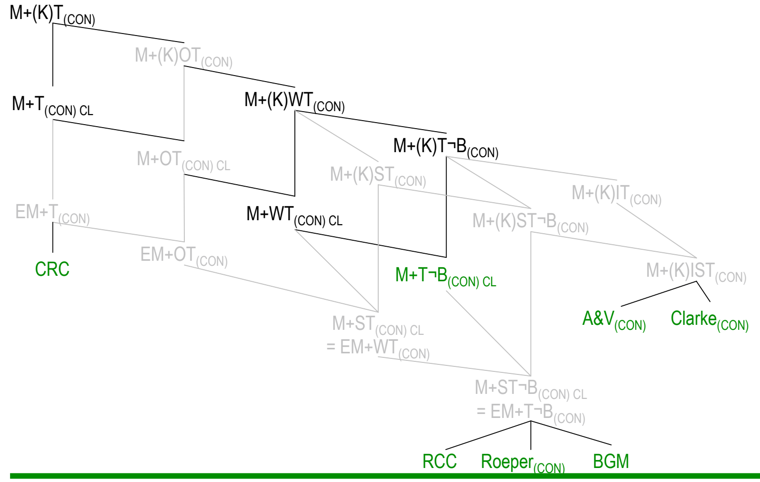
M+FT |= ∀*x y z* [m-fus(*x*; *y*, *z*) ⇒ t-fus(*x*; *y*, *z*)]

M+FT |= **M+T$_{CL}$**

CM+FWT |= CT

M+CT¬B$_{CL}$ |= **M+FT**

Selection from the first map



Mereological and topological sum and intersection principles

M+T$_{CL}$: ∀*x y* [*y* <• *x* ⇒ E(*y*, *x*)]

M+WT: ∀*x y* [E(*x*, *y*) ⇒ *x* <• *y*]

M+T¬B: ∀*x* ∃*y* [*y* $<_i$ *x*]

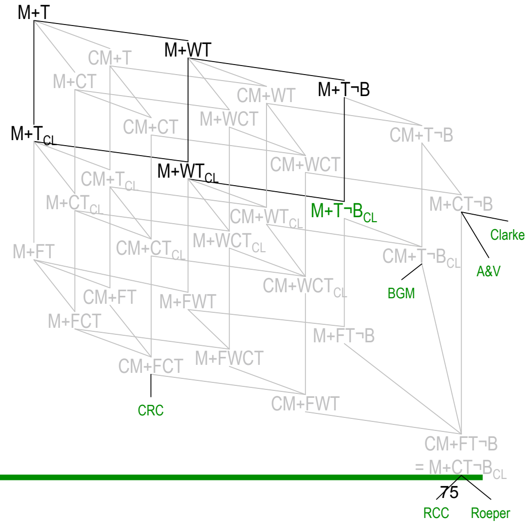
M+FT: ∀x y z [m-fus(x; y, z) ⇒ t-fus(x; y, z)]

CM: ∀y z ∃x [m-fus(x; y, z)]

CM: ∀y z [y ○ z ⇒ ∃x [m-isct(x; y, z)]]

CT: ∀y z ∃x [t-fus(x; y, z)]

CT: ∀y z [S(y, z) ⇒ ∃x [t-isct(x; y, z)]]



Mereological and topological sum and intersection principles

M+T$_{CL}$: ∀*x y* [*y* <• *x* ⇒ E(*y*, *x*)]

M+WT: ∀*x y* [E(*x*, *y*) ⇒ *x* <• *y*]

M+T¬B: ∀*x* ∃*y* [*y* $<_i$ *x*]

M+FT: ∀x y z [m-fus(x; y, z) ⇒ t-fus(x; y, z)]

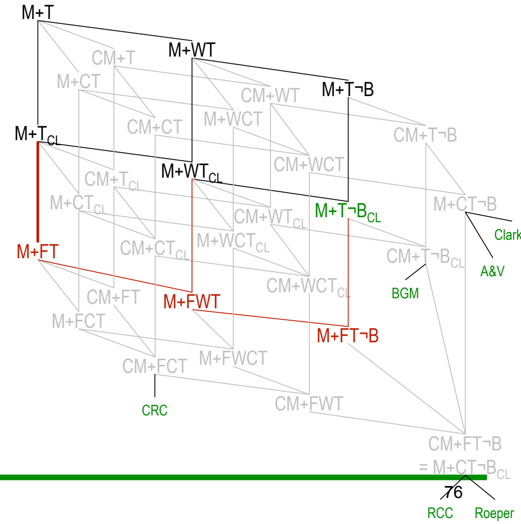
CM: ∀y z ∃x [m-fus(x; y, z)]

CM: ∀y z [y ○ z ⇒ ∃x [m-isct(x; y, z)]]

CT: ∀y z ∃x [t-fus(x; y, z)]

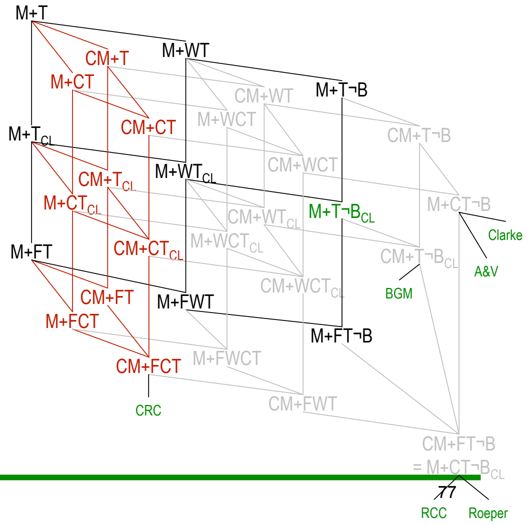
CT: ∀y z [S(y, z) ⇒ ∃x [t-isct(x; y, z)]]

M+FT |= M+T$_{CL}$



Mereological and topological sum and intersection principles

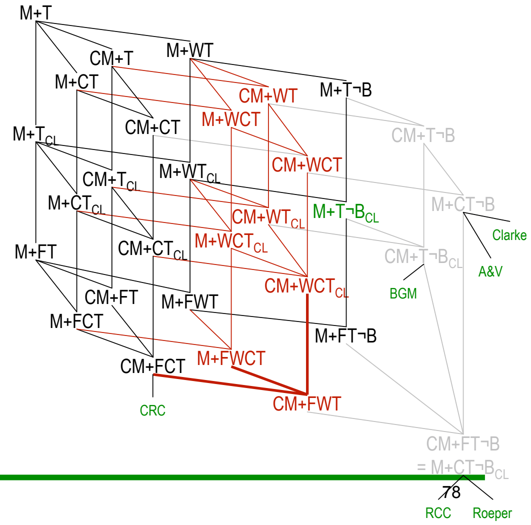
M+T$_{CL}$: $\forall x\, y\, [y \lessdot x \Rightarrow E(y, x)]$
M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x \lessdot y]$
M+T¬B: $\forall x\, \exists y\, [y <_i x]$
M+FT: $\forall x\, y\, z\, [\text{m-fus}(x; y, z) \Rightarrow \text{t-fus}(x; y, z)]$
CM: $\forall y\, z\, \exists x\, [\text{m-fus}(x; y, z)]$
CM: $\forall y\, z\, [y \circ z \Rightarrow \exists x\, [\text{m-isct}(x; y, z)]]$
CT: $\forall y\, z\, \exists x\, [\text{t-fus}(x; y, z)]$
CT: $\forall y\, z\, [S(y, z) \Rightarrow \exists x\, [\text{t-isct}(x; y, z)]]$



Mereological and topological sum and intersection principles

M+T$_{CL}$: $\forall x\, y\, [y \lessdot x \Rightarrow E(y, x)]$
M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x \lessdot y]$
M+T¬B: $\forall x\, \exists y\, [y <_i x]$
M+FT: $\forall x\, y\, z\, [\text{m-fus}(x; y, z) \Rightarrow \text{t-fus}(x; y, z)]$
CM: $\forall y\, z\, \exists x\, [\text{m-fus}(x; y, z)]$
CM: $\forall y\, z\, [y \circ z \Rightarrow \exists x\, [\text{m-isct}(x; y, z)]]$
CT: $\forall y\, z\, \exists x\, [\text{t-fus}(x; y, z)]$
CT: $\forall y\, z\, [S(y, z) \Rightarrow \exists x\, [\text{t-isct}(x; y, z)]]$

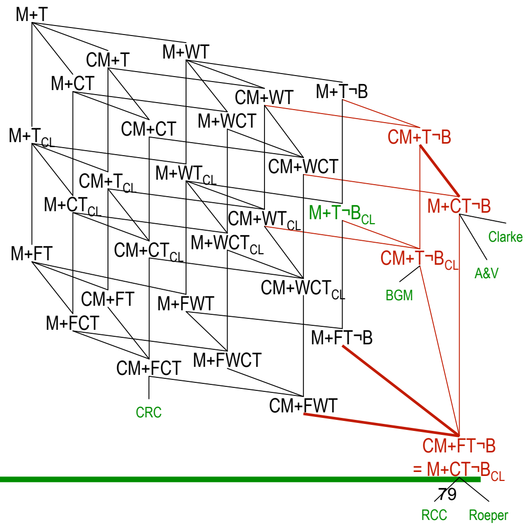
CM+FWT |= CT



Mereological and topological sum and intersection principles

M+T$_{CL}$: $\forall x\, y\, [y \lessdot x \Rightarrow E(y, x)]$
M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x \lessdot y]$
M+T¬B: $\forall x\, \exists y\, [y <_i x]$
M+FT: $\forall x\, y\, z\, [\text{m-fus}(x; y, z) \Rightarrow \text{t-fus}(x; y, z)]$
CM: $\forall y\, z\, \exists x\, [\text{m-fus}(x; y, z)]$
CM: $\forall y\, z\, [y \circ z \Rightarrow \exists x\, [\text{m-isct}(x; y, z)]]$
CT: $\forall y\, z\, \exists x\, [\text{t-fus}(x; y, z)]$
CT: $\forall y\, z\, [S(y, z) \Rightarrow \exists x\, [\text{t-isct}(x; y, z)]]$

M+CT¬B |= CM
M+CT¬B$_{CL}$ |= M+FT



Conclusion

Avoid

Discrete spaces

- if you want topological distinctions

Theories of closed regions

- if you want interiors in non-discrete spaces

General existence of interiors

- if you want theories of closed regions or mereological extensionality

Theories of closed regions with inner parts

- if you want connected space with atoms
- if you want to guarantee topological complements

Allow

Space to be connected

- i.e. that only mereologically universal regions are isolated

Topologically universal regions that are not mereologically universal

- if you want to guarantee mereological complements without guaranteeing inner parts