Modeling in Knowledge Representation: The Parthood Relation

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Doctorate Course

2006-2007

Homework till Monday

- How do your research interest relate to the topics of the course ?
- In which way do parts / the notion of 'part' play a role in your area ?
- Which kinds of parts / wholes play a role ?
- Is the spatial structure important (e.g. spatial connectedness) ?
- Where do temporal notions come up?
- Are temporal instants sufficient for your needs? Where can time periods be important ?

3-Mereotopology

Doctorate Course

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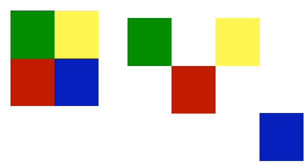
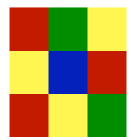
Why topological notions?

Mereology alone

- does not have any notion of 'whole', 'integrity', 'connectedness'
- does not distinguish spatially coherent (one-piece) entities from spatially disconnected (multi-piece) entities
- does not distinguish inner parts from boundary parts

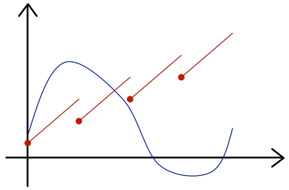




Mathematical Contribution

Point-set Topology

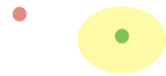
- establishment at the beginning of the 20th century
- general theory for describing continuity
- continuity: predictability of the behavior of a function at the boundary from the behavior in the interior of a set
- Points are undefined primitive objects that serve as basis (carrier) for the structure
- Sets of points define and exhibit the topological structure



Relations between Points and Sets

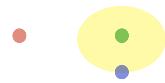
Set Theory

- Points are element of the set
- or not.



Topology

- Points are interior to the set (then they are elements of the set)
- exterior to the set (then they are not elements of the set)
- or boundary points of the set
    - boundary points can be elements of the set
    - or not.



Why Mereotopology ?

Formalizing common-sense knowledge

- proved to be much harder than formalized expert knowledge
- is based on a common-sense ontology
- that includes objects of every day live
- rather than sets.

Mathematics (Topology)

- uses set theory to represent real world problems
- provides sophisticated tools for expert reasoning.

Topology: A Reminder

Definition

- A *topology* on a carrier set D is (given by) a set $\mathcal{T} \subseteq 2^D$ having the following properties
    - $D, \emptyset \in \mathcal{T}$
    - $\forall S \subseteq \mathcal{T} [\bigcup S \in \mathcal{T}]$
    - $\forall X, Y \in \mathcal{T} [X \cap Y \in \mathcal{T}]$
- The elements of $\mathcal{T}$ are called *open sets.*

Simple Examples

- $\{D, \emptyset\}$ (trivial topology)
- 2^D (discrete topology)

Topology Based on a Metric Space

Let $\langle D, d\rangle$ be a metric space, ($d: D \times D \to \mathbb{R}$).

- The *open ball* (in D) of diameter $0 < \varepsilon \in \mathbb{R}$ around x is defined as:

  $B_d(x, \varepsilon) = \{y \in D \mid d(x, y) < \varepsilon\}$
- Let $\mathcal{B}_d$ be the set of all open balls around any point of D, i.e.

  $\mathcal{B}_d = \{B_d(x, \varepsilon) \mid 0 < \varepsilon \in \mathbb{R}, x \in D\}$
- Then the set $\mathcal{T}_d = \{\bigcup M \mid M \subseteq \mathcal{B}_d\}$ is a topology for D.

Examples

- $D = \mathbb{R}$; open balls are open intervals
- $D = \mathbb{R}^2$; open balls are discs without bounding circle line
- $D = \mathbb{R}^3$; open balls are spheres without surface

Topology Induced by a Metric

Example: Let $\langle D, d\rangle$ be a metric space.

- Let $\mathcal{T}_d = \{\bigcup M \mid M \subseteq \mathcal{B}_d\}$.
- Show
  - $D, \varnothing \in \mathcal{T}_d$
  - $\forall S \subseteq \mathcal{T}_d\ [\bigcup S \in \mathcal{T}_d]$
  - $\forall X, Y \in \mathcal{T}_d\ [X \cap Y \in \mathcal{T}_d]$
- Show and take advantage of:

  $\forall B \in \mathcal{B}_d\ [B = \bigcup\{X \in \mathcal{B}_d \mid X \subseteq B\}]$

Points in Topology

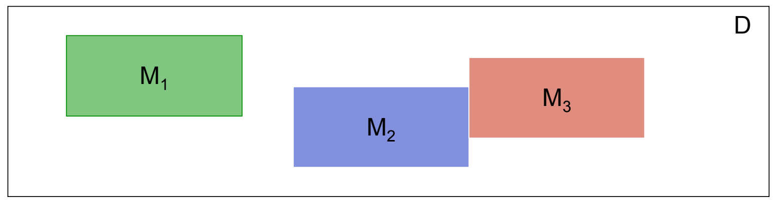
Let $\mathcal{T}$ be a topology on D, $M \subseteq D$ and $p \in D$.

- interiorPoint$_{\mathcal{T}}$(p, M) $\Leftrightarrow_{def}$ $\exists X \in \mathcal{T}[p \in X \wedge X \subseteq M]$
- exteriorPoint$_{\mathcal{T}}$(p, M) $\Leftrightarrow_{def}$ $\exists X \in \mathcal{T}[p \in X \wedge X \cap M = \varnothing]$
- limitPoint$_{\mathcal{T}}$(p, M) $\Leftrightarrow_{def}$ $\forall X \in \mathcal{T}[p \in X \Rightarrow X \cap M \neq \varnothing]$
- boundaryPoint$_{\mathcal{T}}$(p, M) $\Leftrightarrow_{def}$ $\forall X \in \mathcal{T}[p \in X \Rightarrow (X \cap M \neq \varnothing \wedge X \not\subseteq M)]$

Functions Mapping Sets in Topology

Let $\mathcal{T}$ be a topology on D and $M \subseteq D$.

- int$_{\mathcal{T}}$(M) = {p ∈ D | interiorPoint$_{\mathcal{T}}$(p, M)}
- ext$_{\mathcal{T}}$(M) = {p ∈ D | exteriorPoint$_{\mathcal{T}}$(p, M)}
- cl$_{\mathcal{T}}$(M) = {p ∈ D | limitPoint$_{\mathcal{T}}$(p, M)}
- bd$_{\mathcal{T}}$(M) = {p ∈ D | boundaryPoint$_{\mathcal{T}}$(p, M)}



$M_4 = M_1 \cup M_2$ $M_5 = M_2 \cup M_3$

General Laws for the Mappings Defined

Let $\mathcal{T}$ be a topology on D and M $\subseteq$ D. (drop index $_{\mathcal{T}}$ for simplicity)

- int(M) $\subseteq$ M
- M $\subseteq$ cl(M)
- ext(M) $\cap$ M = $\varnothing$
- cl(M) = M $\cup$ bd(M)
- int(M) = M \ bd(M)
- bd(M) = cl(M) \ int(M)
- ext(M) = int(M^{-1})
- int(M) = cl(M^{-1})$^{-1}$
- cl(M) = int(M^{-1})$^{-1}$ = ext(M)$^{-1}$
- bd(M) = cl(M) $\cap$ cl(M^{-1})
- bd(M) = bd(M^{-1})
- bd(M) = (int(M) $\cup$ ext(M))$^{-1}$

- cl($\varnothing$) = int($\varnothing$) = ext(D) = $\varnothing$
- cl(D) = int(D) = ext($\varnothing$) = D

- int(int(M)) = int(M)
- cl(cl(M)) = cl(M)
- int(ext(M)) = ext(M)

- int(cl(M)) $\supseteq$ int(M)
- cl(int(M)) $\subseteq$ cl(M)

- bd(int(M)) $\subseteq$ bd(M)
- bd(cl(M)) $\subseteq$ bd(M)

Open and Closed Sets

Let $\mathcal{T}$ be a topology on D and M $\subseteq$ D.

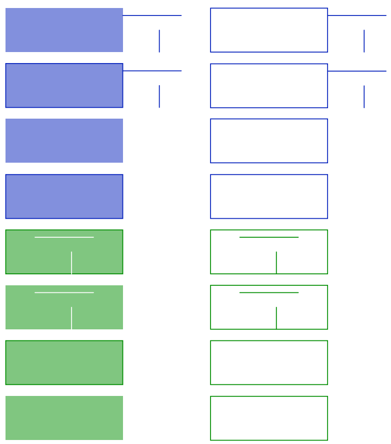
- M is open ($OP_{\mathcal{T}}$(M)) iff M $\in \mathcal{T}$.
- M is closed ($CL_{\mathcal{T}}$(M)) iff $M^{-1} \in \mathcal{T}$.

- $\forall$M $\subseteq$ D [$OP_{\mathcal{T}}$(M) $\Leftrightarrow$ M = $int_{\mathcal{T}}$(M)]
- $\forall$M $\subseteq$ D [$CL_{\mathcal{T}}$(M) $\Leftrightarrow$ M = $cl_{\mathcal{T}}$(M)]
- $\forall$M $\subseteq$ D [$OP_{\mathcal{T}}$($int_{\mathcal{T}}$(M))]
- $\forall$M $\subseteq$ D [$OP_{\mathcal{T}}$($ext_{\mathcal{T}}$(M))]
- $\forall$M $\subseteq$ D [$CL_{\mathcal{T}}$($cl_{\mathcal{T}}$(M))]

Irregular Sets

- cl(int(M_1)) $\subset$ cl(M_1)
- bd(int(M_1)) $\subset$ bd(M_1)

- int(cl(M_2)) $\supset$ int(M_2)
- bd(cl(M_2)) $\subset$ bd(M_2)



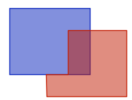
Regular Sets

Let $\mathcal{T}$ be a topology on D and M $\subseteq$ D.

(drop index $_{\mathcal{T}}$ for simplicity)

- M is regular iff cl(int(M)) = cl(M) and int(cl(M)) = int(M).
- M is an open regular set iff OP(M) and int(cl(M)) = int(M).
- M is a closed regular set iff CL(M) and cl(int(M)) = cl(M).

- Unions and intersections of regular sets need not be regular!
- Use (topology-specific) regular union / intersection.



Basic Concepts of Point-Set Topology

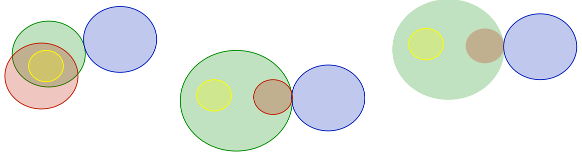
Equivalent axiomatizations based on different primitives

- Open Set
  - a set without a boundary
- Closure
  - mapping a set to the set of all its limit points
- Neighborhoods of a point
  - a neighborhood of a point is extended around the point in all 'directions'

Spatial Structure: Mereotopological Calculi

Basic idea

- (extended) regions are basic entities in the spatial ontology
- topological structure is crucial for spatial structure ($\rightarrow$ qualitative)
- points and boundaries are abstractions from configurations of regions



Topology and Mereology

Beginning of Topology

- Fréchet (1906): Spatial structure of the space of functions
- Hausdorff (1914): General conditions for convergence; Definition of Topology based on 'Neighborhood'
- Kuratowski (1922): Topology based on 'Closure'

Mereology

- Lesniewski (1927–30): Development of Mereology as alternative to set-theory
- Leonard & Goodman (1940): Mereology as a 'Calculus of Individuals'
- Simons (1987): Discussion of alternative axiomatic systems

Mereotopology (1)

- de Laguna (1922): Points and boundaries as abstractions from extended regions
- Whitehead (1929): Proposal of a region-based description of space
- Clarke (1981): Calculus of Individuals based on 'Connection'
- Allen (1981): Time periods in AI
- Randell, Cohn (1989); Vieu (1993): Adaptation of Clarke's calculus for AI
- Egenhofer (1991): Relations between regions based on point-set topology for GIS

Mereotopology (2)

- Randell, Cui, Cohn (1992): Alternative calculus by 'redefining complement'
- Smith (1993): Mereotopology based on 'inner parts'
- Asher & Vieu (1995): geometry of common sense
- Eschenbach & Heydrich (1995): Regions embedded in extensional Mereology
- Varzi (1996): Discussion of alternative axiomatic systems
- Borgo, Guarino, Masolo (1996): pointless theory of space
- Roeper (1997): Region-based Topology
- Masolo & Vieu (1999): Atoms in Mereotopology
- Eschenbach (1999): Closed Region Calculus
- … still an open discussion

Mereotopologies

A large selection of proposals exist

- Whitehead (1929), Clarke (1981), Randell & Cohn (1989), Egenhofer (1991), Randell Cui & Cohn (RCC, 1992), Vieu (1993), Asher & Vieu (A&V, 1995), Roeper (1997), Eschenbach (CRC, 1999), Borgo, Guarino, Masolo (BGM, 1996)

- Is there a common core to the proposals?
- Which approaches can be combined?
- How to choose between 'the proposals' for an application?

How to proceed?

Select terminologies

- Use notation that is neutral regarding the theories
- Identify the common terminological kernel
- Distinguish terms whose definitions differ
    - Mereological terminology (def. based on Part-of $<$)
    - Topological terminology (def. based on contact C)
    - Mereotopological terminology (def. based on $<$ and C)

Identify a list of axioms

- such that every approach can be identified with a subset

Analyse the interrelation between the axioms

Select Terminologies

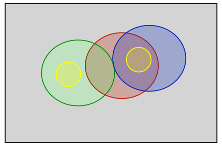
Problems / Obstacles

- Different assumptions on extensionality
  - C-based, ○-based, or both ?
- Different terminology
  - Example: Complement

(D) $x = \text{compl}(y) \Leftrightarrow_{def} \forall u\, [u \bigcirc x \Leftrightarrow \exists w\, [u \bigcirc w \wedge \neg(w \bigcirc y)]]$

(D) $x = \text{compl}(y) \Leftrightarrow_{def} \forall u\, [C(u, x) \Leftrightarrow \exists w\, [C(u, w) \wedge \neg C(w, y)]]$

(D) $x = \text{compl}(y) \Leftrightarrow_{def} \forall u\, [C(u, x) \Leftrightarrow \neg(u <<_i y)] \wedge \forall u\, [u \bigcirc x \Leftrightarrow \neg(u < y)]$



Defining Functions: Complements

(D) $x = \text{compl}(y) \Leftrightarrow_{def} \forall u\, [u \bigcirc x \Leftrightarrow \exists w\, [u \bigcirc w \wedge \neg(w \bigcirc y)]]$

(D) $x = \text{compl}(y) \Leftrightarrow_{def} \forall u\, [C(u, x) \Leftrightarrow \exists w\, [C(u, w) \wedge \neg C(w, y)]]$

(D) $x = \text{compl}(y) \Leftrightarrow_{def} \forall u\, [C(u, x) \Leftrightarrow \neg(u <<_i y)] \wedge \forall u\, [u \bigcirc x \Leftrightarrow \neg(u < y)]$

In order to define a function in standard logic

- the definiens should guarantee (within the assumed theory)
  - uniqueness
  - existence

To allow comparison,

- use unique names
- replace n-ary function-symbol by (n+1)-ary relational symbol

$x = \text{name}_f(y) \Leftrightarrow_{def} \Psi(x, y)$

$\text{id-name}_r(x; y) \Leftrightarrow_{def} \Psi(x, y)$

Mereological Terminology (M):
Mereology: Formalization of the relation of *part* (<)

M: $\forall x\, y\, z\, [z < x \wedge x < y \Rightarrow z < y]$

M: $\forall x\, y\, [x < y \wedge y < x \Rightarrow x = y]$

M: $\forall x\, [x < x]$

$x > y \Leftrightarrow_{def} y < x$

$x << y \Leftrightarrow_{def} x < y \wedge \neg(y < x)$

$x >> y \Leftrightarrow_{def} y << x$

$x \bigcirc y \Leftrightarrow_{def} \exists z\, [z < x \wedge z < y]$

$x \# y \Leftrightarrow_{def} \neg(x \bigcirc y)$

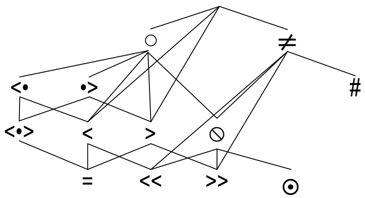
$x \oslash y \Leftrightarrow_{def} x \bigcirc y \wedge \neg(x = y)$

$x \odot y \Leftrightarrow_{def} x \bigcirc y \wedge \neg(x < y) \wedge \neg(y < x)$

$x <\!\bullet\, y \Leftrightarrow_{def} \forall z\, [z \bigcirc x \Rightarrow z \bigcirc y]$

$x \bullet\!> y \Leftrightarrow_{def} y <\!\bullet\, x$

$x <\!\bullet\!> y \Leftrightarrow_{def} x <\!\bullet\, y \wedge y <\!\bullet\, x$



M $\models \forall x\, y\, [x < y \Rightarrow x \bigcirc y]$

M $\models \forall x\, y\, [x <\!\bullet\, y \Rightarrow x \bigcirc y]$

M $\models \forall x\, y\, [x < y \Rightarrow x <\!\bullet\, y]$

Binary Topological Relations

T: $\forall x\, [C(x, x)]$

T: $\forall x\, y\, [C(x, y) \Rightarrow C(y, x)]$

$DC(x, y) \Leftrightarrow_{def} \neg C(x, y)$

$E(x, y) \Leftrightarrow_{def} \forall z\, [C(z, x) \Rightarrow C(z, y)]$

$T \models \forall x\, y\, [E(x, y) \Rightarrow C(x, y)]$

$EQ(x, y) \Leftrightarrow_{def} E(x, y) \wedge E(y, x)$

$S(x, y) \Leftrightarrow_{def} \exists z\, [E(z, x) \wedge E(z, y)]$

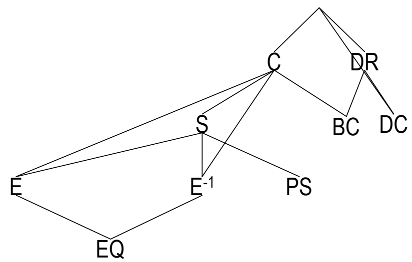
$T \models \forall x\, y\, [S(x, y) \Rightarrow C(x, y)]$

$T \models \forall x\, y\, [E(x, y) \Rightarrow S(x, y)]$

$PS(x, y) \Leftrightarrow_{def} S(x, y) \wedge \neg E(x, y) \wedge \neg E(y, x)$

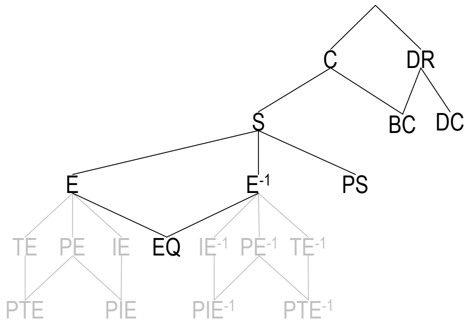
$DR(x, y) \Leftrightarrow_{def} \neg S(x, y)$

$BC(x, y) \Leftrightarrow_{def} C(x, y) \wedge \neg S(x, y)$



Binary Topological Relations

- *connected* (C)
- *disconnected* (DC)
- *enclosure* (E): x encloses y iff everything connected to y is connected to x
- *topological equivalence* (EQ): enclosing and being enclosed
- *superposition* (S): enclosing a common region
- *discrete* (DR): not superposed
- *boundary-connected* (BC): connected without superposition
- *proper superposition* (PS)



Binary Topological Relations

$PE(x, y) \Leftrightarrow_{def} E(x, y) \wedge \neg E(y, x)$

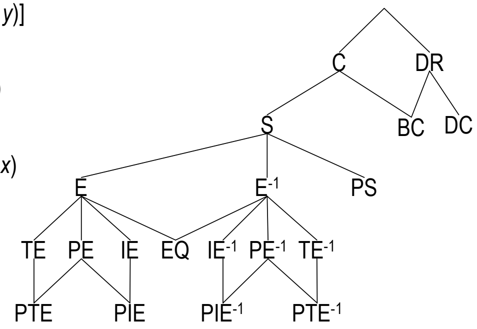
$IE(x, y) \Leftrightarrow_{def} \forall z\, [C(z, x) \Rightarrow S(z, y)]$

$T \models \forall x\, y\, [IE(x, y) \Rightarrow E(x, y)]$

$PIE(x, y) \Leftrightarrow_{def} IE(x, y) \wedge PE(y, x)$

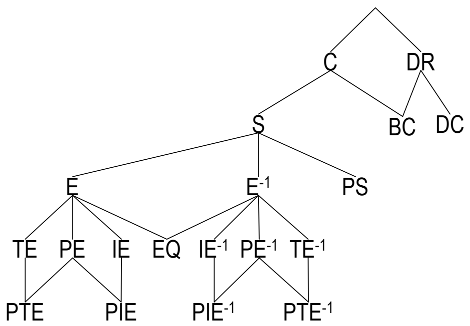
$TE(x, y) \Leftrightarrow_{def} E(x, y) \wedge \neg IE(x, y)$

$PTE(x, y) \Leftrightarrow_{def} TE(x, y) \wedge PE(y, x)$

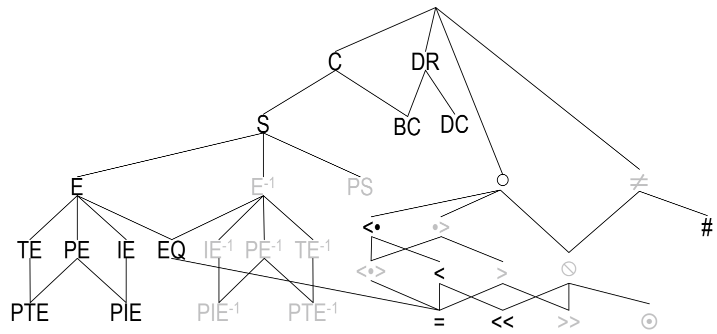


Binary Topological Relations

- *enclosure* (E): x encloses y iff everything connected to y is connected to x
- *proper enclosure* (PE)
- *internal enclosure* (IE): x encloses y internally iff everything connected to y is superposed to x
- *proper internal enclosure* (PIE)
- *tangential enclosure* (TE): enclosure and sharing a boundary-connection
- *proper tangential enclosure* (PTE)
- *minimal connection*, *topological fusion*, *closure*, *interior*, *exterior* …



Mereological and Topological Terminology



Mereotopological Terminology (M+T): Relating Mereology and Topology

Minimal condition

- Parts are (topologically) enclosed

(A) $\forall x\, y\, [x < y \Rightarrow E(x, y)]$

- Overlapping regions are (topologically) superposed and, thus, connected

More specific assumptions

- Replacing Mereology: Identifying the relations part and enclosure

Mereotopological Notions: Refinement of Mereological Relations

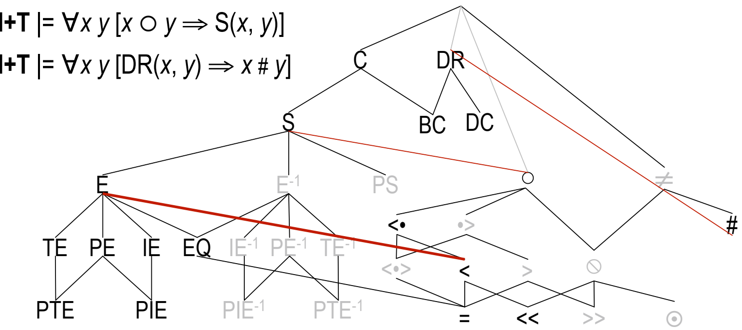
- *External connection* (EC): *Connection* without *overlap*
- *tangential part* ($<_t$), *internal part* ($<_i$)

Mereological and Topological Terminology

M+T: $\forall x\, y\, [x < y \Rightarrow E(x, y)]$

M+T $\models \forall x\, y\, [x \bigcirc y \Rightarrow S(x, y)]$

M+T $\models \forall x\, y\, [DR(x, y) \Rightarrow x \# y]$



Mereological and Topological Terminology

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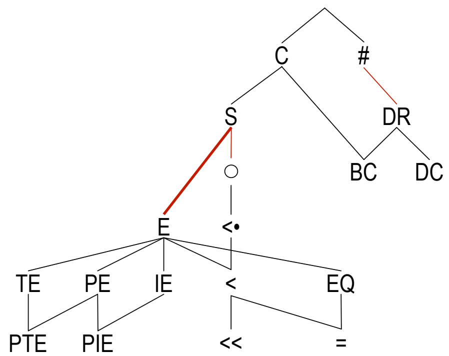


(Selection of) Binary Relations in Mereotopology

M+T: $\forall x\, y\, [x < y \Rightarrow E(x, y)]$

M+T $\models \forall x\, y\, [x \bigcirc y \Rightarrow S(x, y)]$

M+T $\models \forall x\, y\, [DR(x, y) \Rightarrow x \# y]$

$EC(x, y) \Leftrightarrow_{def} C(x, y) \wedge \neg(x \bigcirc y)$

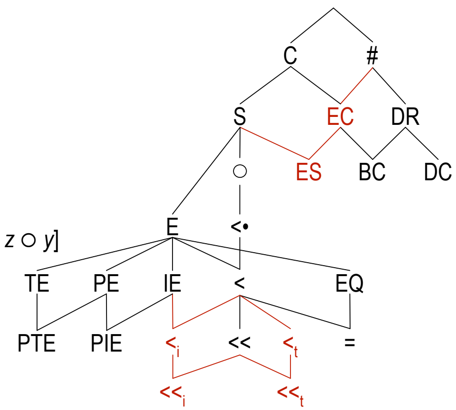
$ES(x, y) \Leftrightarrow_{def} S(x, y) \wedge \neg(x \bigcirc y)$

$x <_i y \Leftrightarrow_{def} x < y \wedge \forall z\, [C(z, x) \Rightarrow z \bigcirc y]$

$x <_t y \Leftrightarrow_{def} x < y \wedge \neg x <_i y$

$x <<_i y \Leftrightarrow_{def} x <_i y \wedge x << y$

$x <<_t y \Leftrightarrow_{def} x <_t y \wedge x << y$



Identify a List of Axioms

Study interactions

Basic Assumptions Collected

The axioms used up to now are

M: $\forall x\, y\, z\, [z < x \wedge x < y \Rightarrow z < y]$

M: $\forall x\, y\, [x < y \wedge y < x \Rightarrow x = y]$

M: $\forall x\, [x < x]$

T: $\forall x\, [C(x, x)]$

T: $\forall x\, y\, [C(x, y) \Rightarrow C(y, x)]$

M+T: $\forall x\, y\, [x < y \Rightarrow E(x, y)]$

Types of Axioms

- Principles of extensionality, supplementation (individuation)
- Interaction between mereological and topological terminology
- Topological and mereological universes
- Existence of open and / or closed regions; divisibility
- A first map
- Atoms and connectedness of space
- Three types of complements and connectedness
- (Further axioms for connectedness)

Extensionality and Interaction of Mereological and Topological Terms

Extensionality

Principles of Extensionality

- Ψ is a formula in which x and z occur freely (but y does not)
- $\forall x\, y\, [x \neq y \Rightarrow \exists z\, [\Psi \Leftrightarrow \neg\Psi[x/y]]]$
- $\forall x\, y\, [\forall z\, [\Psi \Leftrightarrow \Psi[x/y]] \Rightarrow x = y]$

For example, the principle of extensionality in set theory:

- $\forall s1\, s2\, [s1 \neq s2 \Rightarrow \exists a\, [a \in s1 \Leftrightarrow \neg(a \in s2)]]$
- $\forall s1\, s2\, [\forall a\, [a \in s1 \Leftrightarrow a \in s2] \Rightarrow s1 = s2]$

Supplementation in the context of Mereology (employing antisymmetry of <)

- $\forall x\, y\, [\neg(x < y) \Rightarrow \exists z\, [\Psi \wedge \neg\Psi[x/y]]]$
- $\forall x\, y\, [\forall z\, [\Psi \Rightarrow \Psi[x/y]] \Rightarrow x < y]$

Extensionality and the interaction between mereology and topology

ET: Extensional Topology

- $\forall x\, y\, [EQ(x, y) \Rightarrow x = y]$
- **ET** $\models \forall x\, y\, [x \neq y \Leftrightarrow \exists z\, [C(x, z) \Leftrightarrow \neg C(y, z)]]$

EM: Strong supplementation (Extensional Mereology)

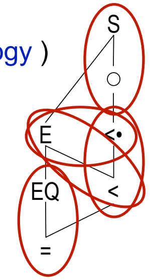
- $\forall x\, y\, [x <\bullet y \Rightarrow x < y]$
- $\forall x\, y\, [\forall z\, [z \circ x \Rightarrow z \circ y] \Rightarrow x < y]$
- **EM** $\models \forall x\, y\, [x \neq y \Leftrightarrow \exists z\, [z \circ x \Leftrightarrow \neg(z \circ y)]]$

M+ST: Strong Mereotopological Interaction

- $\forall x\, y\, [E(x, y) \Rightarrow x < y]$
- $\forall x\, y\, [\forall z\, [C(z, x) \Rightarrow C(z, y)] \Rightarrow x < y]$

M+WT: Weak Mereotopological Interaction

- $\forall x\, y\, [E(x, y) \Rightarrow x <\bullet y]$
- **M+WT** $\models \forall x\, y\, [S(x, y) \Leftrightarrow x \circ y]$



Space of Theories (Extensionality)

ET: $\forall x\, y\, [EQ(x, y) \Rightarrow x = y]$

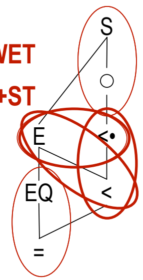
M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x <\bullet y]$

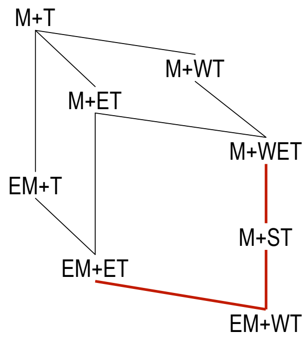
M+ST: $\forall x\, y\, [E(x, y) \Rightarrow x < y]$

EM: $\forall x\, y\, [x <\bullet y \Rightarrow x < y]$

M+ST |= M+WET

EM+WT |= M+ST





Space of Theories (Extensionality)

ET: $\forall x\, y\, [EQ(x, y) \Rightarrow x = y]$

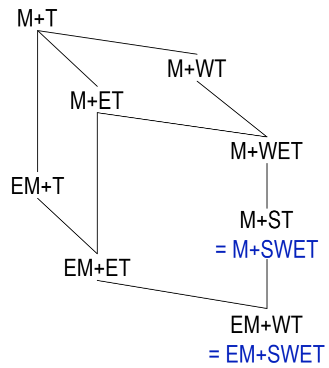
M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x <\bullet y]$

M+ST: $\forall x\, y\, [E(x, y) \Rightarrow x < y]$

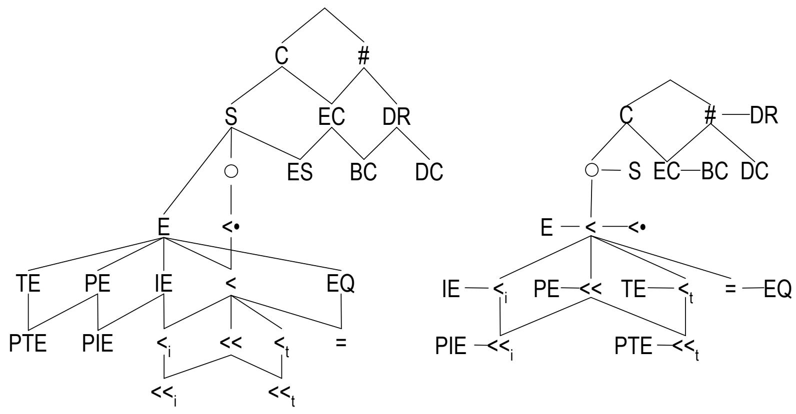
EM: $\forall x\, y\, [x <\bullet y \Rightarrow x < y]$

M+ST |= M+WET

EM+WT |= M+ST



Terminological Simplifications in EM+WT



Space of Theories (Extensionality)

ET: $\forall x\, y\, [EQ(x, y) \Rightarrow x = y]$

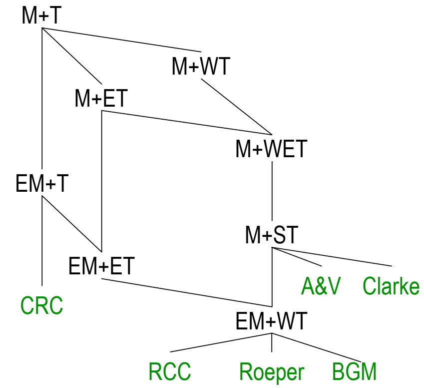
M+WT: $\forall x\, y\, [E(x, y) \Rightarrow x <\bullet\, y]$

M+ST: $\forall x\, y\, [E(x, y) \Rightarrow x < y]$

EM: $\forall x\, y\, [x <\bullet\, y \Rightarrow x < y]$

M+ST |= **M+WET**

EM+WT |= **M+ST**

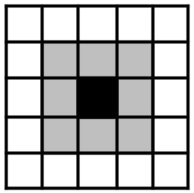


Why not M+ST?

M+ST: $\forall x\, y\, [E(x, y) \Rightarrow x < y]$

Assuming the cells not to have proper parts:

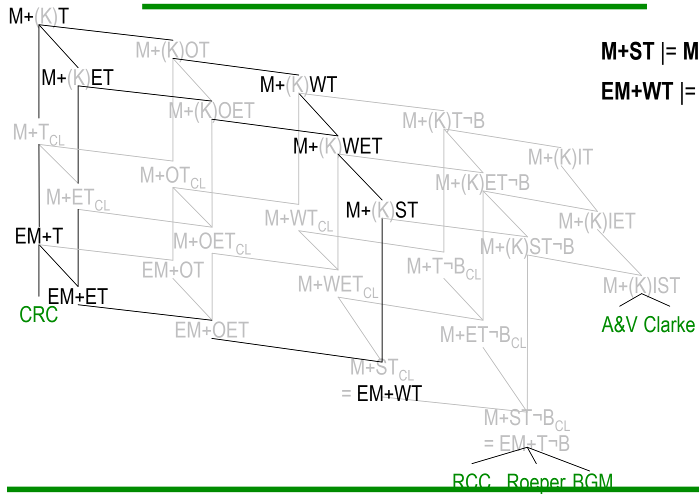
The black square is enclosed by the gray area without being its part.



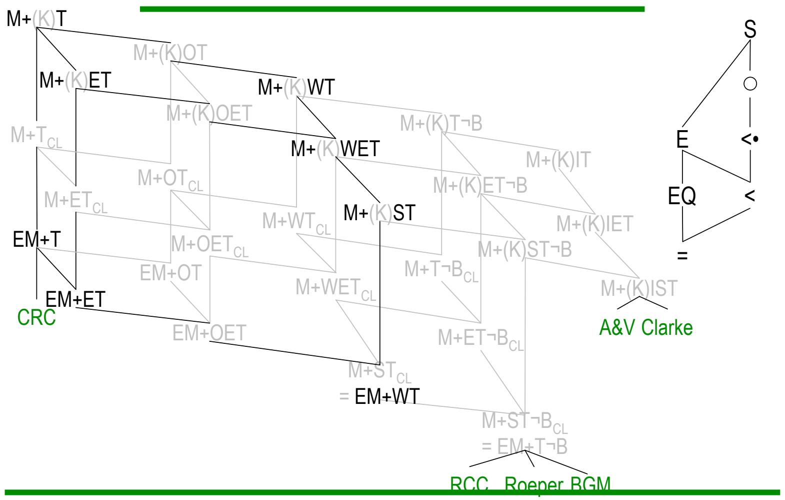
(First Part of) A first map

M+ST |= **M+WET**

EM+WT |= **M+ST**



Principles of extensionality do not contradict each other



Mereologically and Topologically Universal Regions

Universal Regions

Mereologically universal regions

m-univ(x) $\Leftrightarrow_{def}$ ∀y [y ○ x]

M |= ∀x [m-univ(x) ⇔ ∀y [y <• x]]

M |= ∀x [∀y [y < x] ⇒ m-univ(x)]

EM |= ∀x [m-univ(x) ⇔ ∀y [y < x]]

UM: There is a mereologically universal region

∃x [m-univ(x)]

Topologically universal regions

t-univ(x) $\Leftrightarrow_{def}$ ∀y [C(y, x)]

T |= ∀x [t-univ(x) ⇔ ∀y [E(y, x)]]

T |= ∀x [t-univ(x) ⇔ ∀y [S(y, x)]]

UT: There is a topologically universal region

∃x [t-univ(x)]

Universal Regions (2)

Different versions for Mereology and Topology

m-univ(x) $\Leftrightarrow_{def}$ ∀y [y ○ x]

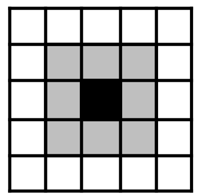
t-univ(x) $\Leftrightarrow_{def}$ ∀y [C(y, x)]

M+T |= ∀x [m-univ(x) ⇒ t-univ(x)]

M+OT: Topologically universal regions are mereologically universal

∀x [t-univ(x) ⇒ m-univ(x)]

- The grey area is topologically universal but not mereologically universal.



Space of Theories (Universal Regions)

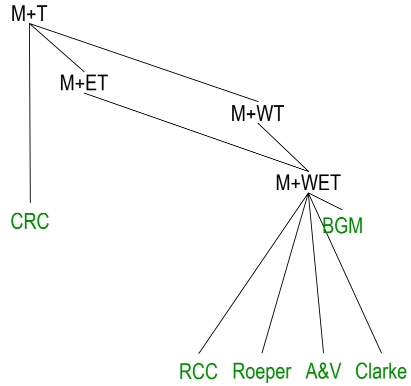
ET: ∀*x y* [EQ(*x*, *y*) ⇒ *x* = *y*]

M+WT: ∀*x y* [E(*x*, *y*) ⇒ *x* <• *y*]

M+UT: ∃*x* [t-univ(*x*)]

M+OT: ∀*x* [t-univ(*x*) ⇒ m-univ(*x*)]

UM+T: ∃*x* [m-univ(*x*)]



Space of Theories (Universal Regions)

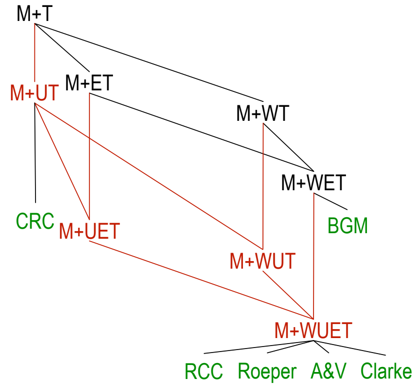
ET: ∀*x y* [EQ(*x*, *y*) ⇒ *x* = *y*]

M+WT: ∀*x y* [E(*x*, *y*) ⇒ *x* <• *y*]

M+UT: ∃*x* [t-univ(*x*)]

M+OT: ∀*x* [t-univ(*x*) ⇒ m-univ(*x*)]

UM+T: ∃*x* [m-univ(*x*)]



Space of Theories (Universal Regions)

ET: ∀*x y* [EQ(*x*, *y*) ⇒ *x* = *y*]

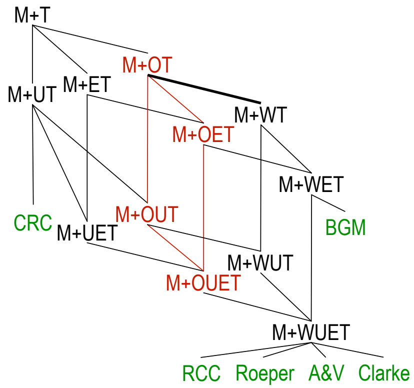
M+WT: ∀*x y* [E(*x*, *y*) ⇒ *x* <• *y*]

M+UT: ∃*x* [t-univ(*x*)]

M+OT: ∀*x* [t-univ(*x*) ⇒ m-univ(*x*)]

UM+T: ∃*x* [m-univ(*x*)]

M+WT |= M+OT



Space of Theories (Universal Regions)

ET: ∀*x y* [EQ(*x*, *y*) ⇒ *x* = *y*]

M+WT: ∀*x y* [E(*x*, *y*) ⇒ *x* <• *y*]

M+UT: ∃*x* [t-univ(*x*)]

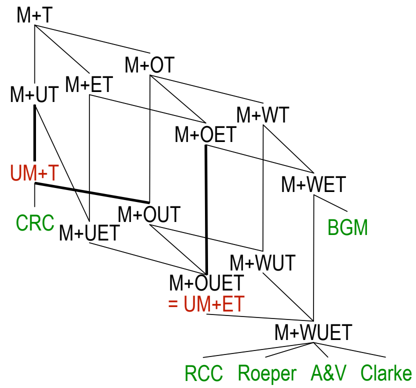
M+OT: ∀*x* [t-univ(*x*) ⇒ m-univ(*x*)]

UM+T: ∃*x* [m-univ(*x*)]

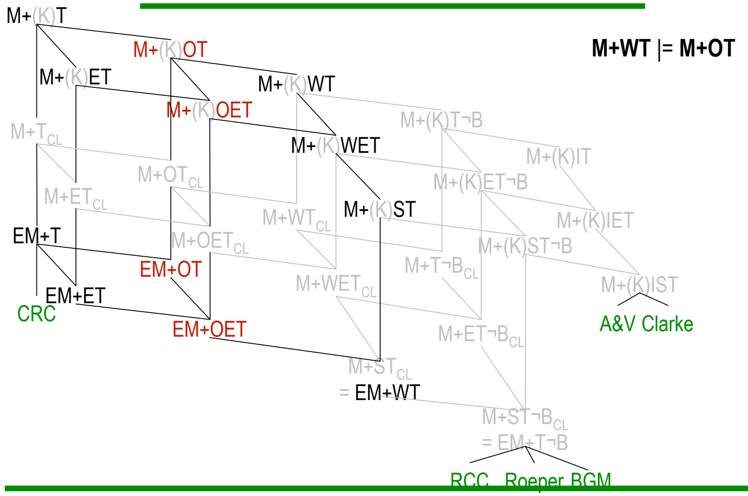
UM+T |= UT

M+OUT |= UM

UM+ET |= M+OT



(Second Part of) A first map



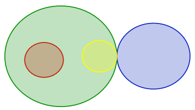
M+WT |= **M+OT**

Closed and Open Regions

Closed and Open Regions

Point-Set Topology

- Closed sets include all their boundaries.
  - Missing connections of closed regions derive from missing parts.

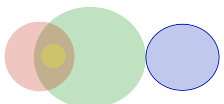


$CL(x) \Leftrightarrow_{def} \forall y\, [y <\!\bullet\, x \Rightarrow E(y, x)]$

M+T $\models \forall x\, [CL(x) \Leftrightarrow \forall y\, [\exists z\, [C(z, y) \wedge \neg C(z, x)] \Rightarrow \exists w\, [w < y \wedge \neg(w \bigcirc x)]]]$

- Open sets do not include their boundaries.
  - All connections derive from sharing parts.
  - There are no external connections to open regions.

$OP(x) \Leftrightarrow_{def} \neg \exists y\, [EC(y, x)]$



Closed Regions and Closures

A region is *closed* iff it encloses all regions it covers.

$CL(x) \Leftrightarrow_{def} \forall y\, [y <\!\bullet\, x \Rightarrow E(y, x)]$

M+T$_{CL}$: All regions are closed

$\forall x\, [CL(x)]$

A *closure* of region y is connected to exactly those regions that are connected to a region y covers.

$cl(x; z) \Leftrightarrow_{def} \forall u\, [C(u, x) \Leftrightarrow \exists y\, [y <\!\bullet\, z \wedge C(u, y)]]$

M+T $\models \forall x\, y\, [cl(x; z) \Rightarrow \forall y\, [y <\!\bullet\, z \Rightarrow E(y, x)]$

M+T $\models \forall x\, [CL(x) \Leftrightarrow cl(x; x)]$

M+KT: All regions have a closure

$\forall y\, \exists x\, [cl(x; y)]$

Theories with Closed Regions and Closures

M+OT: $\forall x$ [t-univ(x) $\Rightarrow$ m-univ(x)]

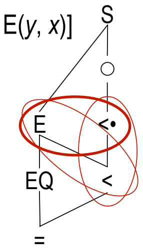
M+WT: $\forall x\, y$ [E(x, y) $\Rightarrow x \lessdot y$]

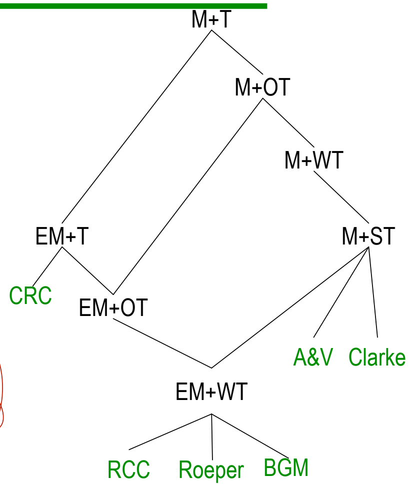
M+ST: $\forall x\, y$ [E(x, y) $\Rightarrow x < y$]

EM: $\forall x\, y$ [$x \lessdot y \Rightarrow x < y$]

M+T$_{CL}$: $\forall x\, y$ [$y \lessdot x \Rightarrow$ E(y, x)]

M+KT: $\forall y\, \exists x$ [cl(x; y)]





Theories with Closed Regions and Closures

M+OT: $\forall x$ [t-univ(x) $\Rightarrow$ m-univ(x)]

M+WT: $\forall x\, y$ [E(x, y) $\Rightarrow x \lessdot y$]

M+ST: $\forall x\, y$ [E(x, y) $\Rightarrow x < y$]

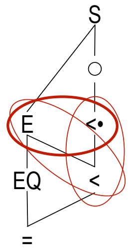
EM: $\forall x\, y$ [$x \lessdot y \Rightarrow x < y$]

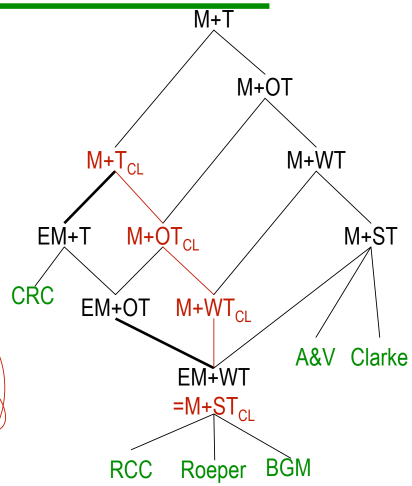
M+T$_{CL}$: $\forall x\, y$ [$y \lessdot x \Rightarrow$ E(y, x)]

M+KT: $\forall y\, \exists x$ [cl(x; y)]

EM+T |= M+T$_{CL}$

M+ST$_{CL}$ |= EM





Theories with Closed Regions and Closures

M+OT: $\forall x$ [t-univ(x) $\Rightarrow$ m-univ(x)]

M+WT: $\forall x\, y$ [E(x, y) $\Rightarrow x \lessdot y$]

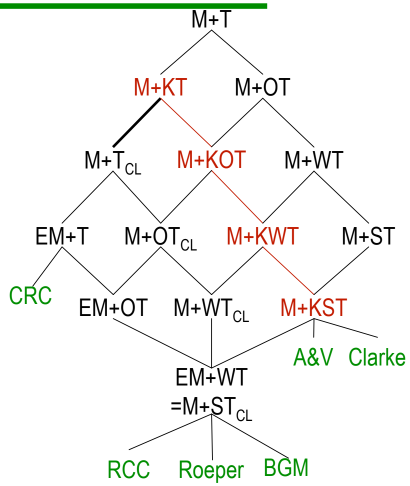
M+ST: $\forall x\, y$ [E(x, y) $\Rightarrow x < y$]

EM: $\forall x\, y$ [$x \lessdot y \Rightarrow x < y$]

M+T$_{CL}$: $\forall x\, y$ [$y \lessdot x \Rightarrow$ E(y, x)]

M+KT: $\forall y\, \exists x$ [cl(x; y)]

M+T$_{CL}$ |= M+KT



Theories with Closed Regions and Closures

M+OT: $\forall x$ [t-univ(x) $\Rightarrow$ m-univ(x)]

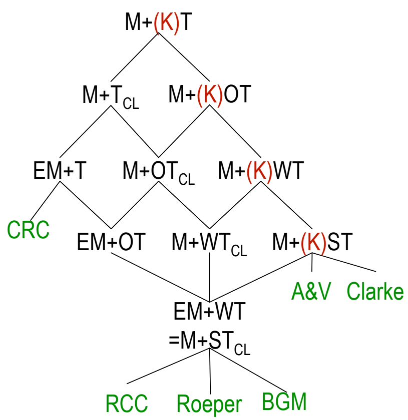
M+WT: $\forall x\, y$ [E(x, y) $\Rightarrow x \lessdot y$]

M+ST: $\forall x\, y$ [E(x, y) $\Rightarrow x < y$]

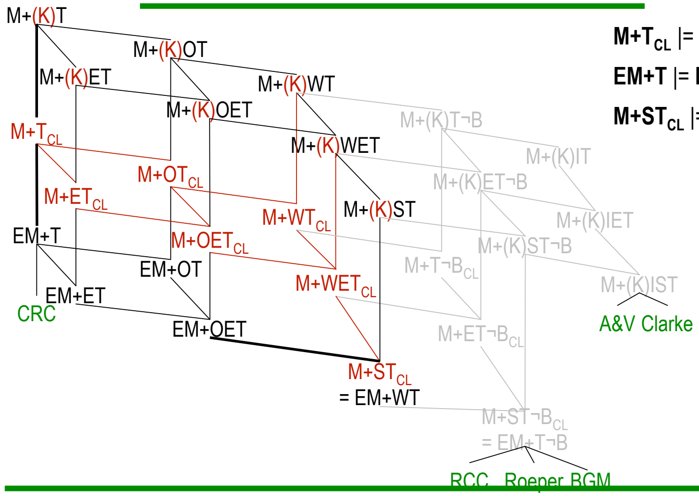
EM: $\forall x\, y$ [$x \lessdot y \Rightarrow x < y$]

M+T$_{CL}$: $\forall x\, y$ [$y \lessdot x \Rightarrow$ E(y, x)]

M+KT: $\forall y\, \exists x$ [cl(x; y)]



(Third Part of) A first map



M+T$_{CL}$ |= M+KT

EM+T |= M+T$_{CL}$

M+ST$_{CL}$ |= EM

Principles of extensionality do not contradict each other
Mereological extensionality stronger than closedness of regions