Modeling in Knowledge Representation: the Parthood Relation

Mereogeometries (Lect. 2 of 2)

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Material we cover today

Lecture 1: Weak Mereogeometries

(1) Lines of sight (Galton)

(2) Occlusion Calculus (Randell et al.)

(3) Convex Hull operator (Cohn)

Lecture 2: Full Mereogeometries

(4) Tarski's geometry of solids.

(5) The problem of the comparison

(6) Some results across theories

Geometrical Primitives:

- Sphere
- Congruence
- Conjugate
- Can Connect
- Closer

A Rough Introduction to MereoGeometry

Motivations
To avoid the commitment to abstract entities (like points, lines and surfaces) in the formalization of space.

Goal
To provide the foundations of geometry in a region-based perspective.

Domain
It may vary. The general constraint is that the elements should provide suitable locations for entities that extends in space (e.g. physical bodies).

Global picture
We are not searching for new/different spaces (although some "different" space might come out as we have seen yesterday).
A mereogeometrical space should capture in a mereological fashion the properties of extended regions.

Geometry of Solids
Tarski, 1929

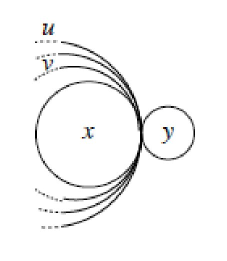
“Some years ago Leṡniewski suggested the problem of establishing the foundations of a *geometry of solids*, understanding by this term a system of geometry destitute of such geometrical figures as points, lines, and surfaces, and admitting as figures only solids – the intuitive correlates of open (or closed) regular sets of three-dimensional Euclidean geometry.”

“The specific character of such a geometry of solids [...] is shown in particular in the law according to which each figure contains another figure as a proper part.”

Basics of GOS

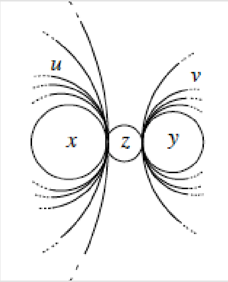
- *Extensional mereology*
  P is the only primitive notion of mereology. “Proper part”, “disjoint”, and “sum” are defined in terms of parthood.
- Axioms for “parthood” and “sum”.
- The notion of *sphere* is the only “geometrical” primitive notion of the geometry.

Tarski's definitions - 1



- Sphere x is externally tangent (ET) to sphere y if (i) x is disjoint from y and (ii) given two spheres u, v containing x as a part and disjoint from y, at least one of them is part of the other.
- Sphere x is internally tangent (IT) to sphere y if (i) x is a proper part of sphere y and (ii) given two spheres u, v containing x as a part and forming part of y, at least one of them is a part of the other.

Tarski's definitions - 2



- Spheres x, y are externally diametrical (ED) to sphere z if (i) each of x, y is externally tangent to z and (ii) given two spheres u, v disjoint from z and such that x is part of u and y is part of v, the sphere u is disjoint from the sphere v.
- Spheres x, y are internally diametrical (ID) to sphere z if (i) each of x, y is internally tangent to z and (ii) given two spheres u, v disjoint from z and such that x is externally tangent to u and y to v, the sphere u is disjoint from the sphere v.

Tarski's definitions - 3

- The sphere x is concentric with the sphere y if one of the following conditions is satisfied: (i) x and y are identical, (ii) x is a proper part of y and, given two spheres u, v externally diametrical to x and internally tangent to y, these spheres are internally diametrical to y, (iii) y is a proper part of x and, given two spheres u, v externally diametrical to y and internally tangent to x, these spheres are internally diametrical to x,
- A point is the class of all spheres which are concentric with a given sphere.

Tarski's definitions - 4

- Points a, b are equidistant from the point c if there exists a sphere x which belongs as element to the point c and is such that: no sphere y belonging as element to the point a or to the point b is a part of x or is disjoint from x
- A solid is an arbitrary sum of spheres.
- The point a is an interior point of the solid y if there exists a sphere x which is at the same time an element of the point a and a part of the solid y.

Axiom

The notions of point and of equidistance of two points from a third satisfy the axioms of ordinary Euclidean geometry of three dimensions.

Tarski's definitions - 4bis

- Points a, b are equidistant from the point c if there exists a sphere x which belongs as element to the point c and is such that: no sphere y belonging as element to the point a or to the point b is a part of x or is disjoint from x
- A solid is an arbitrary sum of spheres. defined or primitive?
- The point a is an interior point of the solid y if there exists a sphere x which is at the same time an element of the point a and a part of the solid y.

Axioms

- *The notions of point and of equidistance of two points from a third satisfy the axioms of ordinary Euclidean geometry of three dimensions.*

- *The class of solids coincides with the class of arbitrary sums of spheres.*

Comparison of Mereogeometries
Borgo, Masolo, 2007

Main Goal:
To compare different mereogeometries in terms of their expressive power.

Framework for the comparison: review of the review...

Recall what has been said about the relationship between:

- Syntax — Semantics
- Theory — Structures
- Interpretation & Models
- Equivalent & Isomorphic Structures
- Soundness — Completeness

Framework for the comparison - 1

First order languages and selected interpretations.
Recall that a relation structure Φ is a sequence $\langle D, R_1, \ldots, R_n \rangle$.

Definitions

- If A is a primitive of theory T, A is explicitly definable in theory T' for a domain D if there exists an expression φ in the language of T' such that the interpretations of P and φ are equivalent in their structures with domain D.

  Example: the primitive P of extensional mereology with standard interpretation is explicitly definable in RCC. We can take:
  $\varphi \equiv \forall z\, (C(z, x) \rightarrow C(z, y))$

- A theory T is a subtheory of T' for domain D if every primitive of T has an explicit definition in T' for that domain.

Framework for the comparison - 2

Definitions (cont'd)

Recall: In general, two theories are equivalent if all the primitives of the first are explicitly definable in the second and viceversa (this is independent of the domain).

This notion leads to the classical notion of equivalence among theories.

We refine this latter as well by making explicit the reference to domains:

- Let T and T' be theories with domains D_i and D_j, respectively. T and T' are *conceptually equivalent* if T is a subtheory of T' and T' is a subtheory of T with respect to both D_i and D_j.

List of Mereogeometries - 1

- $T1$ (Tarski, Bennett) – Geometry of Solids

  Primitives: P, S (where $S(x) =$ "x is a sphere")

  Domain: non-empty regular open subsets of $\mathcal{R}^n$

  Interpretation:

  $P(x,y) \mapsto X \subseteq Y$

  $S(x) \mapsto \exists c \in \mathcal{R}^n, r \in \mathcal{R}^+(X = ball(c,r))$

- $T2$ (Borgo, Guarino, Masolo)

  Primitives: P, SR, CG (where $SR(x) =$ "x is a simple region", $CG(x,y) =$ "x is congruent to y")

  Domain: non-empty regular open subsets of $\mathcal{R}^n$ with finite diameter

  Interpretation:

  $SR(x) \mapsto strong - connected(X)$

  $CG(x,y) \mapsto congruent(X,Y)$

List of Mereogeometries - 2

- $T3$ (Nicod)

  Primitives: P, $Conj$ (where $Conj(x, y; z, w) =$ "x, y and z, w are conjugate")

  Domain: non-empty regular closed connected subsets of $\mathcal{R}^n$

  Interpretation: $Conj(x, y; z, w) \mapsto \exists a, b, c, d(a \in X \wedge b \in Y \wedge c \in Z \wedge d \in W \wedge dist(a, b) = dist(c, d))$

- $T4$ (De Laguna, Donnelly)

  Primitives: $CCon$ (where $CCon(x, y, z) =$ "x can connect both y and z")

  Domain: non-empty regular closed connected subsets of $\mathcal{R}^n$ with finite diameter

  Interpretation: $CCon(x, y, z) \mapsto dist(Y, Z) \leq diam(X)$

List of Mereogeometries - 3

- $T5$ (van Benthem, Aurnague, Vieu, Borillo)

  Primitives: C, $Closer$ (where $Closer(x, y, z)$ = "x is closer to y than to z")

  Domain: non-empty regular subsets of $\mathcal{R}^n$

  Interpretation:

  $C(x, y) \mapsto X \cap Y \neq \emptyset$

  $Closer(x, y, z) \mapsto dist(Y, X) < dist(Z, X))$

- $T6$ (Cohn, Bennett, Gooday, Gotts)

  Primitives: C, $Conv$ (where $Conv(x, y)$ = "x is the convex hull of y")

  Domain: non-empty regular open subsets of $\mathcal{R}^n$

  Interpretation:
  $Conv(x, y) \mapsto convex(X) \wedge Y \subseteq X \wedge \neg\exists Z\,(convex(Z) \wedge Y \subseteq Z \wedge Z \subset X)$

Domains

STRUCT.	DOMAIN	DOMAIN DESCRIPTION *non-empty regular regions and...*	NATURAL ENV.
Φ_α	$D_\alpha = D_O = \{X \subseteq R^n \mid X \neq \varnothing \wedge [X]^\circ = X\}$	*open*	**T1**, **T6**
Φ_β	$D_\beta = \{X \in D_O \mid diam(X) < +\infty\}$	*open and finite*	**T2**
Φ_γ	$D_\gamma = \{X \in D_O \mid Conx(X)\}$	*open and connected*	
Φ_δ	$D_\delta = \{X \in D_O \mid Conx(X) \wedge diam(X) < +\infty\}$	*open, finite and connected*	
Φ_ε	$D_\varepsilon = D_C = \{X \subseteq R^n \mid X \neq \varnothing \wedge [X^\circ] = X\}$	*closed*	**T3**
Φ_ζ	$D_\zeta = \{X \in D_C \mid diam(X) < +\infty\}$	*closed and finite*	
Φ_η	$D_\eta = \{X \in D_C \mid Conx(X)\}$	*closed and connected*	
Φ_θ	$D_\theta = \{X \in D_C \mid Conx(X) \wedge diam(X) < +\infty\}$	*closed, finite and connected*	**T4**

Note that we do not have the domain of $T5$ in the list. This is not considered since the adopted technique cannot handle this domain.

Embedding $T5$ into $T4$

What remains to do is to verify when the primitives of a theory can be defined within another theory in the list. Here we sketch one direction between $T4$ and $T5$: we show that $T5$ (which has primitives C and $Closer$) is a subtheory of $T4$ (primitive $CCon$).

- First, $C(x, y)$ is definable in terms of $Closer$ itself: start with $\neg\exists z Closer(x, z, y)$, note that this does not work for infinite regions, add some further definitions to get the right formula...
- Define $Closer(z, x, y)$ in $T5$ by $\exists w\,(CCon(w, z, x) \wedge \neg CCon(w, z, y))$

  Proof: We need to show that the interpretations of $Closer(z, x, y)$ and that of $\exists w\,(CCon(w, z, x) \wedge \neg CCon(w, z, y))$ are equivalent.

  - The first is: $dist(Z, X) < dist(Z, Y)$
  - The second is $\exists W\,(dist(Z, X) \leq diam(W) \wedge \neg dist(Z, Y) \leq diam(W))$
  - The latter is equivalent to $\exists W\,(dist(Z, X) \leq diam(W) < dist(Z, Y))$
  - The result follows (with some comment...)

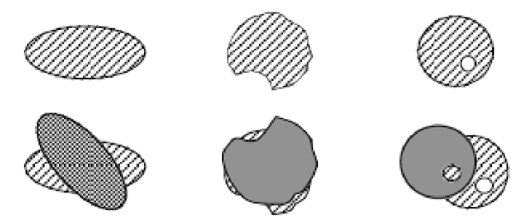
Result of the comparison

Theorem

- Theories $T1, T2, T3, T4, T5$ are equivalent in all the listed domains;
- Theory $T6$ is a subtheory of the others for all the listed domains;
- Theories $T1, T2, T3, T4$ are conceptually equivalent.

About "in mereological fashion"...

A mereogeometrical space should capture <u>in a mereological fashion</u> the properties of extended regions.



$$S(x) =_{\text{def}} SR(x) \wedge \forall y\,((CG(x,y) \wedge PO(x,y)) \rightarrow SR(x-y))$$