Brief Intro to Modal Logic

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Overview

Content

- Propositional Logic
- Worlds
- Propositional Modal Logic
- Syntax and Semantics
- First-Order Modal Logic

Review

First-Order Logic

Review: First-Order Logic

- **Syntax**: The logic and non-logic symbols of the language, the rules for constructing well-formed expressions (formulas) of the logic.
- **Semantics**: The meanings of the atomic symbols of the logic, and the rules for determining the meanings of well-formed expressions.
- **Proof Theory**: The rules for determining new formulas (theorems of the logic) from those given.

Content

A simpler logic: propositional logic

Syntax

Symbols of propositional logic

- symbols for propositional connectives:
  $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\leftrightarrow$ (if and only if)
- logical symbols for propositional constants: $\top, \bot$
- non-logical symbols for propositional constants: p, q, r...
- separation symbols (parentheses): (,)

A language of propositional logic is an *enumerable set of non-logical propositional constants*.

Examples for your intuition

(1) “Cats are white <u>and</u> dogs are black”

(2) “<u>If</u> the chair is broken <u>then</u> the chair is broken”

(3) “The chair is broken <u>if and only if</u> John is standing”

(1) $C \wedge D$

(2) $A \rightarrow A$

(3) $B \leftrightarrow J$

Can you guess the meaning of the following?

(3) $A \rightarrow (B \wedge D)$

(4) $A \vee \neg A$

Formulas

Given a language L, a formula (*of* or *in* L) is any finite sequence of symbols.

The formulas which "receive meaning" are called well formed formulas (wff)
The set of formulas of L is the set obtained by the following rules:

(1) Every (logical and non logical) symbol for propositional constants is a formula
(2) If A is a formula, then $\neg A$ is a formula
(3) If $\circ$ is a binary connective, A and B formulas, then $(A \circ B)$ is a formula

Examples of formulas: A, $A \rightarrow (A \rightarrow B)$, $((C \wedge B) \wedge \neg(C \vee \bot))$

Order of the connectives

As in first-order logic, to simplify the notation, we use an ordering on the connectives (from left to right)

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

Thus, formula $((C \wedge B) \rightarrow (\neg(C \vee A)))$
can be simplified to

$$C \wedge B \rightarrow \neg(C \vee A)$$

The binary connectives are associative to the right

$A \rightarrow B \rightarrow C \rightarrow D$ is equivalent to $A \rightarrow (B \rightarrow (C \rightarrow D))$

Interpretation (models) of propositional logic

Interpretations (or *models* or *structures*) provide the information to associate meaning to formulas in the language.

An interpretation of a language specifies

(1) the truth-value of each non-logical propositional constant in the language.

Obviously, the interpretation depends on the language since different languages may have different propositional constants

Model (interpretation, structure)

We write $\mathcal{M} \models \varphi$
to mean
"model $\mathcal{M}$ satisfies formula φ"

Analogously, $\mathcal{M} \not\models \varphi$
to mean
"interpretation $\mathcal{M}$ does not satisfy formula φ"

NOTE: in the previous lecture we called it "interpretation" ($\mathcal{I}$)

Sentence true in a model $\mathcal{M}$

Let $\mathcal{M}$ be a model for the propositional language L.

(1) $\mathcal{M} \models \top$ and $\mathcal{M} \not\models \bot$

(2) If A is a non-logical constant, then
$\mathcal{M} \models A$ iff $\mathcal{M}$ assigns value T to formula A

(3) $\mathcal{M} \models \neg A$ iff $\mathcal{M} \not\models A$

(4) $\mathcal{M} \models A \wedge B$ iff $\mathcal{M} \models A$ and $\mathcal{M} \models B$

(5) $\mathcal{M} \models A \vee B$ iff $\mathcal{M} \models A$ or $\mathcal{M} \models B$

(6) $\mathcal{M} \models (A \rightarrow B)$ iff $\mathcal{M} \not\models A$ or $\mathcal{M} \models B$

(7) $\mathcal{M} \models (A \leftrightarrow B)$ iff both $\mathcal{M} \models A$ and $\mathcal{M} \models B$
or both $\mathcal{M} \not\models A$ and $\mathcal{M} \not\models B$

Example

$$\mathcal{M} \models A \wedge (B \vee \neg C)$$

(on the board)

Valid, Satisfiable, Contingent

A formula A of the language L is *valid* if and only if it is true in all the models (interpretations, structures) of L. In this case, we write

$$\models A$$

to indicate that A is true no matter the model we put on the left of the '$\models$' sign.

A set of formulas Γ is said to be *satisfiable* if and only if there exists a model (interpretation, structure) $\mathcal{M}$ such that $\mathcal{M} \models A$ for each formula A in Γ.

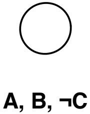
A set of formulas Γ is said to be *contingent* if and only if there exists a model (interpretation, structure) $\mathcal{M}$ such that $\mathcal{M} \models A$ for each formula A in Γ and there exists another model (interpretation, structure) $\mathcal{M}'$ such that $\mathcal{M}' \not\models A$ for each formula A in Γ.

Content

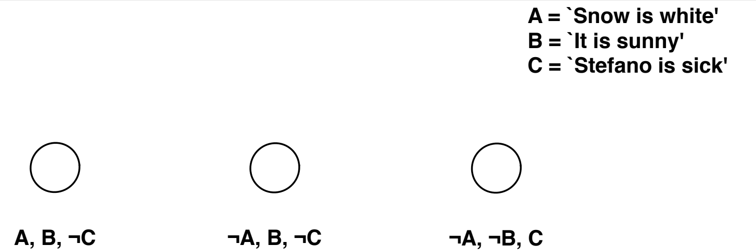
Worlds

It might be otherwise - 1

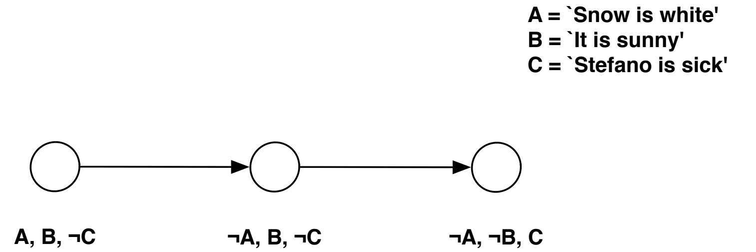
A = `Snow is white'
B = `It is sunny'
C = `Stefano is sick'



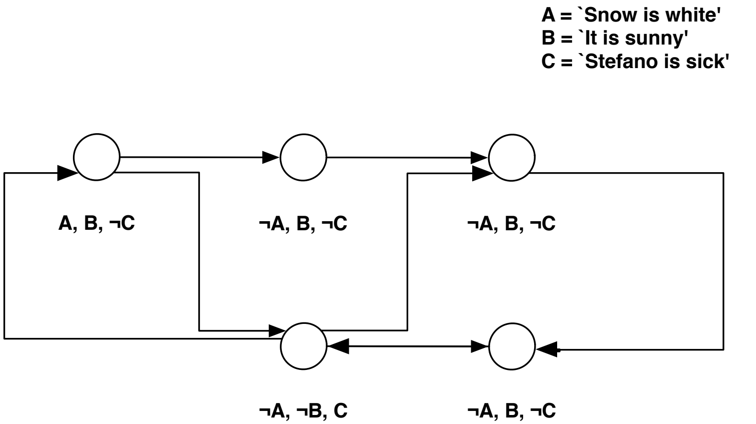
It might be otherwise - 2



It might be otherwise - 3



It might be otherwise - 4



Syntax

The language of propositional modal logic is that of modal logic augmented with two new (logical) operators:

- $\Box$ (necessary, at any time, always in the future...)
- $\Diamond$ (possibly, sometimes, sometimes in the future...)

A language of propositional modal logic is an *enumerable set of non-logical propositional constants*.

Examples for your intuition

(1) "Cats are always white and dogs are sometimes black"

(2) "Necessarily cats are white and dogs are black"

(3) "If the chair is broken sometimes in the future then John will be sometimes standing"

(1) $\Box C \wedge \Diamond D$

(2) $\Box(C \wedge D)$

(3) $\Diamond B \rightarrow \Diamond J$

Can you guess the meaning of the following?

(3) $A \rightarrow \Box B$

(4) $\Box\Diamond A$

Formulas and ordering

The set of formulas of propositional modal logic is the set obtained by the following rules:

(1) Every propositional constant is a formula
(2) If A is a formula, then $\neg A$ is a formula
(3) If $\circ$ is a binary connective, A and B formulas, then $(A \circ B)$ is a formula
(4) If A is a formula, then $\Box A$ and $\Diamond A$ are formulas

Ordering of the connectives

$$\Box, \Diamond, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

Interpretation (model) of propositional modal logic

An interpretation (or model) is a triple $\mathcal{M} = \langle W, R, Int \rangle$ where

(1) W is a set of worlds (or states)

(2) R is a binary relation on W (the accessibility relation)

(3) Int associates a truth-value of each non-logical propositional constant in each world, i.e., $Int(w, A)$= T or F for each constant A of the language and $w \in W$.

For $w \in W$, we write $\mathcal{M}, w \models \varphi$
to mean
"model $\mathcal{M}$ satisfies formula φ in world w"

Analogously, we write $\mathcal{M}, w \not\models \varphi$
to mean
"model $\mathcal{M}$ does not satisfy formula φ in world w"

Sentence true in model $\mathcal{M}$

Let $\mathcal{M}$ be a model for the propositional modal language L.

(1) For all $w \in W$, $\mathcal{M}, w \models \top$ and $\mathcal{M}, w \not\models \bot$

(2) If A is a non-logical constant, then $\mathcal{M}, w \models A$ iff $Int(w, A)$ = T

(3) $\mathcal{M}, w \models \neg A$ iff $\mathcal{M}, w \not\models A$

(4) $\mathcal{M}, w \models A \wedge B$ iff $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$

(5) $\mathcal{M}, w \models A \vee B$ iff $\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$

(6) $\mathcal{M}, w \models (A \rightarrow B)$ iff $\mathcal{M}, w \not\models A$ or $\mathcal{M}, w \models B$

(7) $\mathcal{M}, w \models (A \leftrightarrow B)$ iff both $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$
or both $\mathcal{M}, w \not\models A$ and $\mathcal{M}, w \not\models B$

(8) $\mathcal{M}, w \models \Box A$ iff for all $w' \in W$ with $R(w, w')$ we have $\mathcal{M}, w' \models A$

(9) $\mathcal{M}, w \models \Diamond A$ iff there exists $w' \in W$ such that $R(w, w')$
for which $\mathcal{M}, w' \models A$

Example

$$\mathcal{M}, w \models \Diamond B \vee \neg\Box C$$

(on the board)

Valid, Satisfiable, Contingent

A formula A of the language L is *valid* if and only if it is true in all the worlds of all the models (interpretations, structures) of L. In this case, we write

$$\models A$$

to indicate that A is true no matter the model and the world we put on the left of the '$\models$' sign.

A set of formulas Γ is said to be *satisfiable* if and only if there exists a model (interpretation, structure) $\mathcal{M}$ and a world w in it such that $\mathcal{M}, w \models A$ for each formula A in Γ.

Modalities for FOL

A quick look at first-order modal logic

Language and Formulas

The symbols are those of first order logic. We simply add modalities $\Box$ and $\Diamond$ as logical symbols.

The set of well formed formulas is the set obtained by the usual rules plus the following that we have already seen:

If A is a formula, then $\Box A$ and $\Diamond A$ are formulas

Examples of formulas:

$\Box\forall x Loves(x, John)$

$\Diamond\exists x Loves(John, x)$

$\forall x\exists y\Diamond Loves(y, x)$

Models - 1

A model is a quadruple $\mathcal{M} = \langle W, D, R, Int \rangle$ where:

(1) W is a set of worlds (non-empty set)

(2) D is a domain (non-empty set)

(3) R is a binary relation on W (the accessibility relation)

(4) Int associates at each world a denotation for each constant in the language, that is, an element of the domain that carries that constant as a name.

(5) Int associates at each world a function from D^n to D for each function of arity n in the language.

(6) Int associates at each world a set of n-tuples for each predicate of arity n in the language.

Models - 2

For $w \in W$, we write $\mathcal{M}, w \models \varphi$
to mean
"model $\mathcal{M}$ satisfies formula φ in world w"

Analogously, we write $\mathcal{M}, w \not\models \varphi$
to mean
"model $\mathcal{M}$ does not satisfy formula φ in world w"

Sentence true in model $\mathcal{M}$

Let $\mathcal{M}$ be a model for the propositional modal language L.

(1) $\mathcal{M}, w \models \top$ and $\mathcal{M}, w \not\models \bot$

(2) If A is an atomic formula of type $P(t_1, \ldots, t_n)$, then
$\mathcal{M}, w \models P(t_1, \ldots, t_n)$ iff $t_1^{Int_w}, \ldots, t_n^{Int_w} \in P^{Int_w}$

(3) If A is an atomic formula of type $t_1 = t_2$ then
$\mathcal{M}, w \models t_1 = t_2$ iff $t_1^{Int_w} = t_2^{Int_w}$

(4) $\mathcal{M}, w \models \neg A$ iff $\mathcal{M}, w \not\models A$

(5) $\mathcal{M}, w \models A \wedge B$ iff $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$

(6) $\mathcal{M}, w \models A \vee B$ iff $\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$

(7) $\mathcal{M}, w \models (A \rightarrow B)$ iff $\mathcal{M}, w \not\models A$ or $\mathcal{M}, w \models B$

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or both $\mathcal{M}, w \not\models A$ and $\mathcal{M}, w \not\models B$

Sentence true in model $\mathcal{M}$

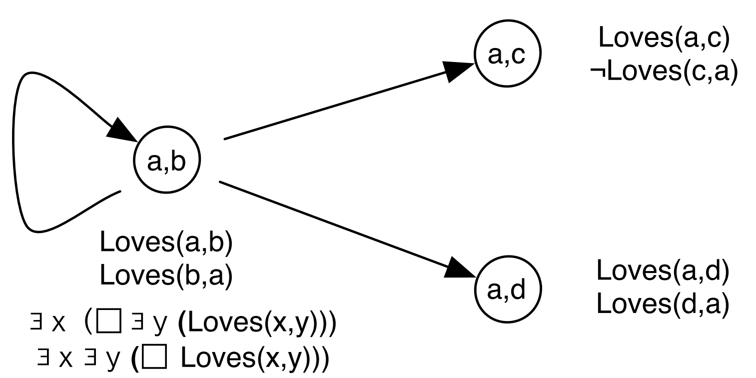
(9) $\mathcal{M}, w \models \Box A$ iff for all $w' \in W$ such that $R(w, w')$ we have $\mathcal{M}, w' \models A$

(10) $\mathcal{M}, w \models \Diamond A$ iff there exists $w' \in W$ such that $R(w, w')$ for which $\mathcal{M}, w' \models A$

(11) $\mathcal{M}, w \models \forall x A$ iff for all $d \in D$ we have $\mathcal{M}, w \models A_{\{x=d\}}$

(12) $\mathcal{M}, w \models \exists x A$ iff there exists a $d \in D$ for which $\mathcal{M}, w \models A_{\{x=d\}}$

Example


a,b
a,c
a,d
Loves(a,c)
¬Loves(c,a)
Loves(a,b)
Loves(b,a)
Loves(a,d)
Loves(d,a)
∃ x (□ ∃ y (Loves(x,y)))
∃ x ∃ y (□ Loves(x,y)))

References

Here there are some references you can use for both lectures. Recall that we have seen really too little of proof theory, beside that, you should be able to read the rest of the material.

- ▷ entries in the Stanford Encyclopedia of Philosophy:
  "Classical Logic" (sections 1,2,4 and parts of 5)
  "Modal Logic" (sections 1-6 and 13)
  "Temporal Logic" (sections 1,2)
  http://plato.stanford.edu/contents.html
- ▷ Beginning Logic by E.J.Lemmon, Chapman & Hall/CRC Publisher (1965, second edition 1987)
- ▷ Chapter "Elementary Predicate Logic" (sections 1 and 2) by W. Hodges in the Handbook of Philosophical Logic (1983) – fairly hard