Kinds of First-order Modal Logic

Nicola Guarino
(thanks to John Mylopoulos)

Modalities

- Most important ways of interpreting modal operators:
  - Temporal modality: “X is always true”
  - Epistemic modality: “X is believed to be true”
  - Deontic modality: “X ought to be true”
  - Alethic modality: “X is necessarily true”

Conditions on the accessibility relation

It makes sense to constrain the links between possible worlds. Here are some possibilities:

- *Reflexivity*: For every world w, $w \rightarrow w$, which basically says that the actual world is also possible
- *Transitivity*: For all worlds w_1, w_2, w_3 if $w_1 \rightarrow w_2$ and $w_2 \rightarrow w_3$ then $w_1 \rightarrow w_3$
- *Seriality (Continuation)*: For every w there is another world w_1 such that $w \rightarrow w_1$

Seriality means that there are no dead ends. Actually, we don't need seriality if we have reflexivity.

Axiom schemes related to properties of accessibility relation

1. No conditions
2. Seriality (D): $\Box P \Rightarrow \Diamond P$
3. Reflexivity (T): $\Box P \Rightarrow P$
4. Transitivity (4): $\Box P \Rightarrow \Box\Box P$
5. Symmetry (B): $P \Rightarrow \Box\Diamond P$
6. Symmetry and transitivity (5): $\Diamond P \Rightarrow \Box\Diamond P$

Temporal Modality

- Here we interpret as possible world as the status of a certain system at a certain time point, and the accessibility relation as a *temporal precedence* relation between worlds. So $w_i \rightarrow w_{i+1}$ means that world w_{i+1} temporally follows w_i.
- The modal operator is now "henceforth", or "from now on".
- The Kripke semantics is therefore:

In any world w_i "henceforth X" iff X is true in all worlds w_i, w_{i+1}, w_{i+2}, …

Axioms for Temporal Logics

- In any world w_t (world at time t) "henceforth T" for every tautology T (B1)
- At any time t, "henceforth P" and "henceforth P ⇒ Q" implies "henceforth Q" (B2)
- At any time t, "henceforth P" implies P (B3)
- At any time t, "henceforth P" implies "henceforth henceforth P" (B4)

Epistemic modality: belief

- Possible worlds are *epistemic states*.
- To say that Maria believes X is to say that in all possible worlds compatible with Maria's current epistemic state, X is true.
- So the accessibility relation is interpreted in terms of (consistent) knowledge transition.
- Standard Modal Logics for Belief make some pretty strong assumptions:
  - In all possible worlds, Maria believes T for every tautology T (B1)
  - Maria believes all logical consequences of her beliefs, I.e., if she believes X and also X ⇒ Y, then she believes Y (B2)
- These axioms may be a bit unrealistic in the sense that they make believers God-like mathematicians!

More Axioms for Belief

- The usual definition of knowledge (vs belief) is that knowledge is true belief.
- We can represent this with an axiom that states (forcing reflexivity of links)

  "if Maria believes X then X" (B3)
- Transitivity of links implies introspection: "If Maria believes X, then she believes that she believes X" (B4)
- Modal Logics that uphold transitivity (hence axiom (B4)) are known as *Doxastic Logics* (Gr. Δοξαζω= Believe); Doxastic logics don't include (B3)
- *Epistemic Logics* adopt axioms (B1') - (B4'), where the modality is not "believe", but rather "know".

Deontic Logics

- Let's forget reflexivity for now, and focus on transitivity instead.
- In Deontic Logics, we interpret $w_1 \rightarrow w_2$ to mean that w_2 is a better world than w_1.
- Continuation is important here: To speak about what ought to be done, we need to have an idea of worlds that are better …
- Changing "believe" into "obligation", we adopt axioms (B1), (B2), (B4), but also

  If in w "X is obligatory" then not "not-X is obligatory" (D)
- Not "not-X is obligatory" is equivalent to "X is permissible"

Alethic modality

- Now we link every possible world to every other possible world, I.e., for all w_i, w_j, $w_i \rightarrow w_j$ (that is, accessibility relation is a *total relation*)
- Now

  "necessarily P" iff P is true in every possible world (Kripke")
- We no longer need to specify in which world "necessarily P" is true, since if it is in one, it is in all.
- Axioms (B1) - (B4) can now be rephrased in the obvious way.
- Since now the accessibility relation is also symmetric, we have: "possibly P" implies necessarily "possibly P" (B5)

Three basic modal logic systems

Modal Logic **K**

- The most basic modal logic is a weak logic of necessity called **K** (after Kripke)
- **K** includes Propositional Logic, along with the modal operators $\Box$P (= necessarily P) and $\Diamond$p (= possibly P).
- The axioms of **K** have as follows:
  - *Necessitation rule*: If A is a theorem, so is $\Box$A
  - *Distribution axiom*: $\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$
- A and B are metavariables refering to formulas in **K**.
- Necessitation means that all theorems of Propositional Logic are necessary.

⇔ As discussed earlier, $\Diamond p =_{def} \neg\Box\neg A$

S4 and S5

- In **K**, you can't prove
  ($\Box A \Rightarrow A$) (M)
  even though this is clearly desirable for necessity (…but not for obligation!)
- Many logicians recommend two more axioms to govern the nesting of necessity/possibility:
  ($\Box A \Rightarrow \Box\Box A$) (4)
  ($\Diamond A \Rightarrow \Box\Diamond A$) (5)
- **S4** is the Logic that results from adding to **K** (M) and (4). In **S4**, any string of $\Box$ ($\Diamond$) can be replaced by one.
- **S5** is the Logic that results from adding to **K** (M) and (5). In **S5**, any string of $\Box$/$\Diamond$ can be replaced by the last operator in the string.

Quantifiers in Modal Logics

- Quine has argued that it doesn't make sense to quantify over possible worlds …
- Key problem: what's the quantification domain? Two ways:
  - Fixed-domain -- "possibilistic"
  - World-relative -- "actualistic" the individuals over which you quantify change from world to world
- For fixed-domain approaches, we just adopt everything from Propositional Logic, plus the Barcan formula:
  $\forall x \Box A \Rightarrow \Box \forall x A$ (BF)
- Unfortunately, fixed-domain approaches are controversial because they interprets existence as possible existence.
- To deal with this, we may want to add E meaning "actually exists"

De dicto vs. de re reference

- Consider:
  - necessarily, all bachelors are unmarried *(de dicto* reference to bachelors*)*
  - all bachelors are necessarily unmarried *(de re* reference to bachelors)
- De dicto reference: quantifier *inside* the modal scope
- De re reference: quantifier *outside* the modal scope

References

- M. Fitting and R. Mendelsohn, *First Order Modal Logic*, Kluwer 1998.
- *Stanford Encyclopedia of Philosophy*, Modal Logic.
- [Hughes68] Hughes, G. and Cresswell, M., *An Introduction to Modal Logic*, London, Methuen, 1968.