Mereogeometries: a Semantic Comparison

Stefano Borgo and Claudio Masolo

Technical Report 02/01, LADESB-CNR
Padova, Italy, January 2001

Questo lavoro   stato condotto nell'ambito dell'attivit del gruppo di ricerca
Modellazione concettuale e Ingegneria della Conoscenza
del LADSEB-CNR

LADSEB-CNR
Corso Stati Uniti 4
I-35127 PADOVA (PD)
e-mail: mbox@ladseb.pd.cnr.it
fax: +39 049-829.5763
tel: +39 049-829.5702

Abstract

We compare several geometrical theories based on mereology (*mereogeometries*) in a unified framework. Most theories in this area lack in formalization and this prevents any systematic logical analysis. We overcome this problem by isolating a common domain in R^n and, selecting natural interpretations, we use this framework to show several interdependencies among primitive relations of these theories. We conclude that, for dimension $n \leq 3$ and with some additional assumptions, most of the theories considered are equivalent in the provided interpretation.

Sommario

In questo lavoro vengono confrontate in un ambiente unificato molte geometrie basate sulla mereologia (*mereogeometrie*). La maggior parte di queste teorie sono deficitarie dal punto di vista della caratterizzazione formale e ci preclude un'analisi logica sistematica. Questo problema viene evitato identificando in R^n un dominio comune a tutte le teorie e, fissate le interpretazioni naturali delle primitive, si utilizza questo ambiente unificato per evidenziare alcuni legami sussistenti tra le relazioni primitive delle varie teorie. Viene mostrato come, per dimensioni $n \leq 3$, introducendo alcune ipotesi aggiuntive sul dominio, molte delle teorie considerate sono equivalenti nell'interpretazione fissata.

Mereogeometries: a Semantic Comparison

Stefano Borgo
Dept. of Mathematics, Computer Science
Indiana University, Swain East 227
Bloomington, IN 47405-5701
USA
stborgo@indiana.edu

Claudio Masolo
Dept. of Electronics and Computer Science
Via Gradenigo, 6A
35100 Padova
Italy
masolo@ladseb.pd.cnr.it

Abstract

We compare several geometrical theories based on mereology (*mereogeometries*) in a unified framework. Most theories in this area lack in formalization and this prevents any systematic logical analysis. We overcome this problem by isolating a common domain in R^n and, selecting natural interpretations, we use this framework to show several interdependencies among primitive relations of these theories. We conclude that, for dimension $n \leq 3$ and with some additional assumptions, most of the theories considered are equivalent in the provided interpretation.

1 Introduction

Since the eighteen century, the need for rigor in mathematics has suggested a more formal approach to classical areas including Euclidean geometry. This has lead to investigations into the ontological principles explicitly and implicitly assumed in mathematical fields.

The subsequent development of modern geometry shows that the formalism could not help in deciding ontological matters when interdefinable primitive relations are involved.

Despite the fact that Lobacevskij published his approach to geometry as early as 1829, we had to wait for Whitehead [1929], Nicod [1962], and de Laguna [1922] before critical analysis of *geometrical entities* was seriously developed. These authors, refusing the classical commitment to points, lines, and planes, consider extended regions (bodies, solids, or volumes) as primitive geometrical entities:

Au lieu de parler de points *et de* relations entre points, *ces g om tries parlent de* volumes *et de* relations entre volumes°; *de m me qu'ailleurs on nomme volume une certaine classe de points, on nomme ici point, inversement, une certaine classe de volumes*. ([Nicod, 1962], p.26)

At this point, Euclidean geometry lost its absolute value in mathematics and since then a variety of geometrical theories, differing in their domains or primitive relations, have been introduced. These theories have been studied according to their adequacy to specific tasks at hand. In particular, the authors above were interested in the cognitive plausibility of geometry, which explains why extended regions were taken as basic elements. To phrase is as Nicod did, "nature gives us volumes".

This formalization of new geometries developed at the beginning of the twentieth century, had some drawbacks. The primitives were often only partially characterized [de Laguna, 1922; Whitehead, 1929] and sometimes, as in the work of Tarski and Nicod [Tarski, 1956; Nicod, 1962], only indirectly axiomatized. Tarski considered *parthood* and *being a sphere* as primitives and defined several relations among spheres (e.g. concentricity), relying on the intended interpretation of those primitives. He axiomatized mereology directly and introduced points in the theory as classes of concentric spheres. In this way, he could define equidistance among points using properties of spheres centered on those points and use the well-known axioms of Euclidean geometry to characterize fully the predicate *being a sphere* on which equidistance depends. Nicod took *parthood* and *conjugation* as primitives and assumed all theorems of Euclidean geometry as axioms (providing an intended interpretation for the primitives.) Both these authors were aware of the formal drawbacks of their approaches. Their main goal was to show that extended regions could be taken as the fundamental entities of geometry. However, their indirect approach does not help to isolate those primitive properties that are very important on the cognitive and applicative sides.

Later, Grzegorczyk [1960] and then Clarke [1981; 1985] were able to solve these problems but only limiting themselves to the topological level, considering theories based on *parthood* and *connection* relations (*mereotopologies*). The main reason is that these suffice to define points as second-order entities, which is what they were interested in.

In the last decade, *mereogeometry*, i.e. geometry based on mereology or region-based geometry, has attracted renewed interest and has been studied in mathematics [Gerla, 1994] but also in other domains. One can find mereogeometrical theories in several areas like *formal ontology*, where the notion of parthood and connection are studied in very general terms [Simons, 1987; Casati and

Varzi, 1999]; *cognitive science*, where the formalization of semantics in natural languages and the study of cognitive processes are considered from a qualitative point of view [Aurnague *et al.*, 1997]; and *qualitative spatial representation and reasoning* in AI [Bennett *et al.*, 2000a; Borgo *et al.*, 1996; Dugat *et al.*, 1999; Pratt and Lemon, 1997; Randell and Cohn, 1992; Stock, 1997]. In all these cases, the choice of primitives is determinant to ensure ontological clearness, correspondence between theorical and applicative notions, and to improve computational efficiency. [1]

The increasing request for rich mereogeometries and the variety of application domains where these are being applied, motivate the development of several theories. They are sometimes driven by very specific problems and they end up differing on entities, properties, and principles. The next step in this research area is to develop methods that allow us to compare and classify theories in such a way that differences, expressive powers, and conceptual incompatibilities can be made explicit. In this way, it would be easier to the practitioners to: (i) select a theory which is most apt to the applicative or theoretical tasks at stake, (ii) extend or modify an already available theory, (iii) integrate or exchange results among different theories.

In this paper, we provide an initial answer to this request for comparison. We consider the R^n framework for the study of semantic relationships that can be established among several mereogeometries.

2 A Semantic Strategy of Comparison

Given two first-order theories with a (partial) axiomatization, one may compare them on the syntactic or semantic level in a standard logical approach. In the former case, one exploits the interdefinability of primitive relations between both theories to prove the equivalence of their axiomatics. In the latter case, one compares domains and relations of (classes of) models for the theories. Such systematic analyses are available for mereotopologies only, especially on the syntactic level ([Simons, 1987; Casati and Varzi, 1999; Masolo and Vieu, 1999]) and to some degree on the semantic level ([Biacino and Gerla, 1991; Asher and Vieu, 1995]).

These types of comparison are not really relevant in the case of mereogeometries that are more expressive than mereotopologies, because of lack of rich enough formalization in first order logic. Actually, another kind of comparison has been already used for mereotopologies: Cohn and Varzi [1998] took classical point-set topology as a common framework for comparing the intended interpretations of the theories. We here propose a similar approach for mereogeometries taking R^n as common framework and interpreting the theories' primitives in models embedded in this framework. [2]

Rejecting classical mathematical systems because not ontologically adequate is not incompatible with using them to express and compare models of region-based theories. In general, mereogeometries focus on qualitative relations; they do not aim at capturing "new" notions of space, not even at enriching classical geometry. Indeed, spatial theories adequate to cognitive or applicative tasks are generally expressible in R^n, i.e., Cartesian geometry, which has the advantages of being expressive enough and well understood.

A logician could find this discussion unsatisfactory. In logic, a semantic analysis considers classes of models for a theory, whereas we are selecting only a specific model for each theory. This approach allows us to work with theories having a minimal characterization, and sometimes relying on generic descriptions without going very far into formalization. Only the definitions, with respect to R^n, of the entities in the domain and of the primitive relations among them are necessary. The theories we are considering are relevant because of the intuitions underlying the formalization and not because of their axiomatics. Our task is to provide tools to pinpoint the notions that are more interesting and useful, and understand how/when new notions may be assimilated to those already present in the literature.

A critical point is that we must have an interpretation for each theory. When this is missing or incomplete[3], we must build up one as close as possible to the informal description of the primitives provided by the authors and compatible with the axioms; in short, a natural interpretation. In the case of equivalent interpretations, we choose one that facilitates the comparison from a technical point of view and makes it as clear as possible.

3 Theories in the Comparison

In this section we list the mereogeometries that we will compare, together with (*i*) the basic entities and primitive relations an (*ii*) their intended or natural interpretation in R^n. Relations are indicated by their formal names, the indexes give the arity of the relations: $\mathsf{C}^{(2)}$ stands for *is connected to*, $\mathsf{CCon}^{(3)}$ for *can connect*, $\mathsf{CG}^{(2)}$ for *is congruent to*, $\mathsf{Closer}^{(3)}$ for *is closer to than to*, $\mathsf{Conj}^{(4)}$ for *are conjugate*, $\mathsf{ConvH}^{(2)}$ for *is the convex hull of*, $\mathsf{P}^{(2)}$ for *is part of*, $\mathsf{S}^{(1)}$ for *is a sphere*, $\mathsf{SR}^{(1)}$ for *is a simple region*.

In order to specify the intended or natural interpretations in R^n, we need:

- topological operators: *closure* ([]), *interior* (¡);
- Euclidean distance, *Dist*: $R^n \times R^n \to [0, +\infty)$;

[1] Clearly these three areas are not disjoint, and interdisciplinary studies can contribute to develop theories that are more understandable and reusable.

[2] However, we will not reach the completeness of the result of Cohn and Varzi. These authors were able to characterize all theories that can be interpreted in topological terms. This kind of completeness is much more complex when mereogeometrical theories are considered.

[3] Often these interpretations do not give enough information, for instance about existential conditions of entities.

- standard relations like convex-subspace ($Conv$), connected-subspace[4] ($Conx$), manifold (M), congruent-subspaces ($Congr$), and the following relations definable from these in R^n (lower-case variables stand for points, capital variables for sets):

  $Dist(X, Y) = inf\{Dist(x, y) \mid x \in X \text{ e } y \in Y\}$;
  $Diam(X) = sup\{Dist(x, y) \mid x, y \in X\}$;
  $Ball(c, r) = \{x \mid Dist(x, c) < r\}$, where $r > 0$;
  $Btw(x, y, z)$ iff $Dist(x, y) + Dist(x, z) = Dist(y, z)$;
  $\partial(X) = [X] - X$;

In what follows, $\Im$ is the interpretation function and σ the assignment function. To simplify notations we write X_i instead of $\Im_\sigma(x_i)$ and, for any formula ϕ, $\phi \Rightarrow Phi$ (where Phi is a relation in R^n) instead of $\Im_\sigma(\phi) = Phi$.

Here then are the theories we will focus on:

T1 Theory first presented in [Tarski, 1956] and later developed in [Bennett *et al.*, 2000b]:
$Domain = \{non\text{-}empty\ regular\ open\ subsets\ of\ R^n\}$
$= \{X \subset R^n \mid X \neq \varnothing \wedge [X]¡ = X\}$;
$\mathsf{P}(x, y) \Rightarrow X \subseteq Y$;
$\mathsf{S}(x) \Rightarrow \exists c,r(X = Ball(c, r))$.

T2 Theory presented in [Borgo *et al.*, 1996] and simplified in [Bennett *et al.*, 2000b]:[5]
$Domain = \{non\text{-}empty\ regular\ open\ subsets\ of\ R^n\}$;
$\mathsf{P}(x, y) \Rightarrow X \subseteq Y$;
$\mathsf{SR}(x) \Rightarrow M(X)$;
$\mathsf{CG}(x, y) \Rightarrow Congr(\mathrm{X, Y})$.

T3 Theory given in [Nicod, 1962]: [6]
$Domain = \{non\text{-}empty\ regular\ closed\ connected\ subsets\ of\ R^n\}$
$= \{X \subset R^n \mid X \neq \varnothing \wedge [X¡] = X \wedge Conx(X)\}$;
$\mathsf{P}(x, y) \Rightarrow X \subseteq Y$;
$\mathsf{Conj}(x, y, x', y') \Rightarrow \exists x, y, x', y' (x \in X \wedge y \in Y \wedge x' \in X' \wedge y' \in Y' \wedge Dist(x, y) = Dist(x', y'))$.

T4 Theory introduced in [de Laguna, 1922]: [7]
$Domain = \{non\text{-}empty\ regular\ closed\ connected\ subset\ of\ R^n\ with\ finite\ diameter\}$;
$\mathsf{CCon}(z, x, y) \Rightarrow \exists z, z', x, y\ (z, z' \in Z \wedge x \in X \wedge y \in Y \wedge Dist(z, z') = Dist(x, y))$.

T5 Theory presented in [Aurnague *et al.*, 1997]: [8]
$Domain = \{non\text{-}empty\ regular\ subsets\ of\ R^n\}$;
$= \{X \subset R^n \mid X \neq \varnothing \wedge [X] = [X¡] \wedge X¡ = [X]¡\}$;
$\mathsf{C}(x, y) \Rightarrow X \cap Y \neq \varnothing$;
$\mathsf{Closer}(z, x, y) \Rightarrow Dist(Z, X) < Dist(Z, Y)$.

T6 Theory given in [Cohn, 1995]: [9]
$Domain = \{non\text{-}empty\ regular\ open\ subsets\ of\ R^n\}$;
$\mathsf{C}(x, y) \Rightarrow [X] \cap [Y] \neq \varnothing$;
$\mathsf{ConvH}(x, y) \Rightarrow Conv(X) \wedge Y \subseteq X \wedge \neg\exists Z(Conv(Z) \wedge Y \subseteq Z \wedge Z \subset X)$;

3.1 Choosing the Domain

To compare all the theories in a homogeneous way their models must be based on the same domain.[10] The theories listed above involve three general domains (T3 and T4 include further restrictions):

$D_1 = \{non\text{-}empty\ regular\ open\ subsets\ of\ R^n\}$;
$D_2 = \{non\text{-}empty\ regular\ closed\ subsets\ of\ R^n\}$;
$D_3 = \{non\text{-}empty\ regular\ subsets\ of\ R^n\}$;

All the theories only consider extended entities and do not refer to lower-dimension objects like, for instance, boundaries.[11] In D_3 we find both closed and open regular regions so that the corresponding theory T5 turns out to be richer than the others. However, there are a couple of problematic aspects. From a cognitive point of view, it is not clear which criteria to apply to isolate physical objects associated with open, closed or semi-closed regular regions. On the ontological side, mereological extensionality does not hold in this case; the closure and the interior of a region are both in the domain but not their difference (the boundary itself). It turns out that D_3 gives a peculiar set of entities that increases considerably the complexity of the comparison. Therefore, here we will limit ourselves to the simpler cases given by D_1 and D_2.

Having excluded domain D_3, we notice that for all primitives[12] $\mathsf{R}(x_1, .., x_n)$ and for all the interpretations $\Im$ considered above, $\Im(\mathsf{R})(X_1, .., X_n)$ implies $\Im(\mathsf{R})([X_1], .., [X_n])$ in R^n. We can therefore choose a unique interpretation function that applies to both domains D_1 and D_2 maintaining the same meaning for the primitives. This interpretation $\Im'$ is defined as follows:

- if $\Im(\mathsf{R})(X_1, .., X_n)$ implies $\Im(\mathsf{R})(X_1\ldots, X_n\ldots)$ in R^n, then $\Im'(\mathsf{R}) = \Im(\mathsf{R})$;
- else $\Im'(\mathsf{R})(X_1, .., X_n) = \Im(\mathsf{R})([X_1], .., [X_n])$.

Now we can choose indifferently D_1 or D_2. Let us take D_1, hereafter called D.

---

[4]A connected subspace is often called self-connected in the literature. We will use both terms indifferently.

[5]According to the motivations given by the authors, the theory presented in [Borgo *et al.*, 1996] is explicitly restricted to R^3 but we can easily generalize it to R^n.

[6]The informal interpretation is given at pages 27-28.

[7]The informal interpretation is given at pages 449-450

[8]No intended interpretation for the relation Closer is provided. The interpretation we propose satisfies all the given axioms and, as far as we can tell, it is faithful to their approach.

[9]This theory was originally restricted to R^2. It can be extended to R^n.

[10]Allowing for different domains makes the comparison much more complex from the technical point of view and it is not clear that this would significantly enrich the conceptual analysis.

[11]The approach we present here could be used also to compare the class of theories that admit boundary-like entities (heterogeneous theories as opposed to homogeneous theories like those considered in this paper). However, it is not clear in which terms it is possible to compare heterogeneous vs homogeneous theories.

[12]With the exception of S for which we slightly modify the definition of *Ball*.

Since T5 is defined on D_3, it makes little sense to involve T5 as such in the comparison. However, below we will still consider an interpretation for the primitive relation Closer because of the interesting aspects it presents.

To compare some theories we need additional constraints on D (note that $HP_2 \rightarrow HP_1$):

HP_1: $\forall X, Y \in \mathrm{D}\ (Dist(X, Y) = d \rightarrow \exists x, y\ (x \in [X] \wedge y \in [Y] \wedge Dist(x, y) = d))$;
HP_2: $\forall X \in \mathrm{D}\ (Diam(X) = d < +\infty)$.

HP_1 rules out some (but not all) infinite regions so that, for instance, we cannot find two regions with closest boundaries behaving as asymptotic functions as we move towards infinity in some direction. In particular, this ensures that two regions have zero distance if and only if their closures intersect. HP_2 forces us to consider only finite regions, i.e. regions whose closure is a compact.

3.2 Choosing the Interpretation

Having fixed the domain we now consider the following interpretation for the primitive relations that agrees with the requirements stated above:

$\mathrm{C}(x, y) \Rightarrow [X] \cap [Y] \neq \emptyset$;
$\mathrm{CCon}(z, x, y) \Rightarrow \exists z,z',x,y(z,z' \in [Z] \wedge x \in [X] \wedge y \in [Y] \wedge Dist(z, z') = Dist(x, y))$;
$\mathrm{Closer}(z, x, y) \Rightarrow Dist(Z, X) < Dist(Z, Y)$;
$\mathrm{Conj}(x, y, x', y') \Rightarrow \exists x,y,x',y'(x \in [X] \wedge y \in [Y] \wedge x' \in [X'] \wedge y' \in [Y'] \wedge Dist(x, y) = Dist(x', y'))$;
$\mathrm{ConvH}(x, y) \Rightarrow Conv(X) \wedge Y \subseteq X \wedge \neg\exists Z(Conv(Z) \wedge Y \subseteq Z \wedge Z \subset X)$;
$\mathrm{P}(x, y) \Rightarrow X \subseteq Y$;
$\mathrm{SR}(x) \Rightarrow M(X)$;
$\mathrm{S}(x) \Rightarrow \exists c,r(X = Ball(c, r))$.

On the basis of the domain and the interpretation chosen above, we can demonstrate the following theorems, making explicit the semantics of some defined predicates:

$\mathrm{PP}(x, y) =_{def} \mathrm{P}(x, y) \wedge \neg\mathrm{P}(y, x)) \Rightarrow X \subset Y$;
$\mathrm{O}(x, y) =_{def} \exists z(\mathrm{P}(z, x) \wedge \mathrm{P}(z, y)) \Rightarrow X \cap Y \neq \emptyset$;
$z = x + y \Rightarrow Z = [X \cup Y]$¡;
$\mathrm{SC}(x) =_{def} \forall y,z(x = y + z \rightarrow \mathrm{C}(y, z)) \Rightarrow Conx(X)$;
$\mathrm{IP}(x, y) =_{def} \mathrm{P}(x, y) \wedge \forall z(\mathrm{C}(z, x) \rightarrow \mathrm{O}(z, y)) \Rightarrow X \subset Y \wedge \partial(X) \cap \partial(Y) = \emptyset$;
$\mathrm{SR}(x) =_{def} \forall y,z(x = y + z \rightarrow \exists w(\mathrm{SC}(w) \wedge \mathrm{O}(w, y) \wedge \mathrm{O}(w, z) \wedge \mathrm{IP}(w, x))) \Rightarrow M(X)$;
$\mathrm{MCP}(x, y) =_{def} \mathrm{SC}(x) \wedge \mathrm{P}(x, y) \wedge \neg\exists z(\mathrm{SC}(z) \wedge \mathrm{P}(z, y) \wedge \mathrm{PP}(x, z)) \Rightarrow Conx(X) \wedge X \subseteq Y \wedge \partial(Y \backslash X) = \partial(Y) \backslash \partial(X)$;
$\mathrm{P}(x, y) = \forall z(\mathrm{C}(z, x) \rightarrow \mathrm{C}(z, y)) \Rightarrow X \subset Y$.

We also give the semantics of relations among spheres introduced in [Tarski, 1956] and in [Bennett *et al.*, 2000b]. $\mathrm{SCG}^{(2)}$ stands for *sphere congruence*, $\mathrm{CNC}^{(2)}$ for *concentricity*, $\mathrm{SCG}^{(4)}$ for *pair of sphere congruence* and $\mathrm{BTW}^{(2)}$ for *betweeness*:

$\mathrm{SCG}(x_1,x_2) \Rightarrow \exists c_1,c_2,r\ (X_1 = Ball(c_1, r) \wedge X_2 = Ball(c_2, r))$;
$\mathrm{CNC}(x_1,x_2) \Rightarrow \exists c,r_1,r_2\ (X_1 = Ball(c, r_1) \wedge X_2 = Ball(c, r_2))$;
$\mathrm{SCG}(x_1,x_2,x_3,x_4) \Rightarrow \exists c_1,c_2,c_3,c_4,r_1,r_2\ (X_1 = Ball(c_1, r_1) \wedge X_2 = Ball(c_2, r_2) \wedge X_3 = Ball(c_3, r_1) \wedge X_4 = Ball(c_4, r_2) \wedge Dist(c_1, c_2) = Dist(c_3, c_4))$;
$\mathrm{BTW}(x_1,x_2,x_3) \Rightarrow \exists c_1,c_2,c_3,r_1,r_2,r_3\ (Btw(c_1,c_2,c_3) \wedge X_1 = Ball(c_1,r_1) \wedge X_2 = Ball(c_2,r_2) \wedge X_3 = Ball(c_3,r_3))$.

4 Comparison of the Theories

Fig. 1 shows some semantic links among the interpretations of primitives considered in theories T1-T6.

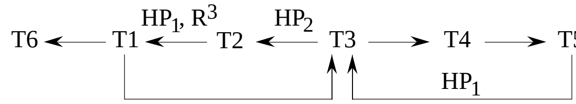

Figure 1. Semantic links among primitives in T1-T6.
(Here R^3 means that a link is verified only for R^n with $n \leq 3$)

An arrow $Ti \rightarrow Tj$ means that it is possible to define the primitive relations of Tj from those of Ti, having proved that the interpretations of the primitives of Tj and those of their interpretations in Ti are equivalent in D. Further assumptions are shown in the labels.

In the following, we give these detailed definitions. The reader can find complete proofs of these semantic equivalences in [Tech. Rep. 2001][13].

T3 → T4 CCon is defined using Conj:

$\mathrm{CCon}(x, y, z) =_{def} \mathrm{Conj}(x, x, y, z)$.

T4 → T5 We first introduce C from which one obtains P, +, and SC as in /3.2:

$\mathrm{C}(x, y) =_{def} \forall z(\mathrm{CCon}(z, x, y))$;
$\mathrm{Closer}(z, x, y) =_{def} \forall w((\mathrm{SC}(w) \wedge \mathrm{CCon}(w, z, y)) \rightarrow \mathrm{CCon}(w, z, x)) \wedge \exists w'(\mathrm{SC}(w') \wedge \mathrm{CCon}(w', z, x) \wedge \neg\mathrm{CCon}(w', z, y))$.

T5 → T3 (assuming HP_1) We first introduce C (relying on the domain D), from which one obtains P, +, SC, and MCP as in /3.2. We define also new relations: $\mathrm{Eq}(z, x, y)$ stands for *z is equidistant from x and y*, $\mathrm{EqD}(x, y, x', y')$ for *x is as close to y as x' is to y'*, $\mathrm{ClD}(x, y, x', y')$ for *x and y are closer each other than x' and y'*, $\mathrm{ClEqD}(x, y, x', y')$ for *x' and y' are not closer each other than x' and* y':

$\mathrm{C}(x, y) =_{def} \neg\exists z(\mathrm{Closer}(x, z, y))$;
$\mathrm{Eq}(z, x, y) =_{def} \neg\mathrm{Closer}(z, x, y) \wedge \neg\mathrm{Closer}(z, y, x)$;
$\mathrm{EqD}(x, y, x', y') =_{def} \exists z(\mathrm{Eq}(x,z,y) \wedge \mathrm{Eq}(x',z,y') \wedge \mathrm{Eq}(z,x,x'))$;
$\mathrm{ClD}(x,y,x',y') =_{def} \exists z(\mathrm{Eq}(x,z,y) \wedge \mathrm{Eq}(x',z,y') \wedge \mathrm{Closer}(z,x,x'))$;
$\mathrm{ClEqD}(x, y, x', y') =_{def} \mathrm{ClD}(x, y, x', y') \vee \mathrm{EqD}(x, y, x', y')$
$\mathrm{Conj}(x, y, x', y') =_{def} \exists a,b,a',b',p_x,p_y,p'_x,p'_y(\mathrm{MCP}(a, x) \wedge \mathrm{MCP}(b, y) \wedge \mathrm{MCP}(a', x') \wedge \mathrm{MCP}(b', y') \wedge \mathrm{C}(p_x, a) \wedge \mathrm{C}(p_y, b) \wedge \mathrm{C}(p'_x, a') \wedge \mathrm{C}(p'_y, b') \wedge \mathrm{ClEqD}(a, b, p_x, p_y) \wedge \mathrm{ClEqD}(a', b', p'_x, p'_y) \wedge \mathrm{EqD}(p_x, p_y, p'_x, p'_y))$.

T3 → T2 (assuming HP_2) In order to define CG from Conj and P, we use an idea already applied in [Borgo *et al.*, 1996], where the authors showed how to define CG from S and P. Since T3, T4, T5 are equivalent under the

[13] Momentarily suppressed to maintain anonymity

assumption HP_1 and since HP_2 is stronger than HP_1, P and all needed relations can be defined using only Conj:[14]

$$\mathsf{EqDiam}(x, y) =_{def} \mathsf{SR}(x) \wedge \mathsf{SR}(y) \wedge$$
$$\forall a,b(\mathsf{P}(a+b, x) \rightarrow \exists a',b'(\mathsf{P}(a'+b',y) \wedge \mathsf{Conj}(a, b, a', b'))) \wedge$$
$$\forall a,b(\mathsf{P}(a+b, y) \rightarrow \exists a',b'(\mathsf{P}(a'+b',x) \wedge \mathsf{Conj}(a, b, a', b')));$$
$$\mathsf{S\text{-}SR}(x) =_{def} \forall y(\mathsf{MCP}(y, x) \rightarrow$$
$$\neg\exists z(\mathsf{MCP}(z, x) \wedge \neg z = y \wedge \mathsf{EqDiam}(z, y))));$$
$$\mathsf{S\text{-}CGR}(x, y) =_{def} \mathsf{S\text{-}SR}(x) \wedge \mathsf{S\text{-}SR}(y) \wedge$$
$$\forall u,v((\mathsf{MCP}(u, x) \wedge \mathsf{MCP}(v, x) \wedge \neg u = v)$$
$$\rightarrow \exists u',v'(\mathsf{MCP}(u', y) \wedge \mathsf{MCP}(v', y) \wedge \mathsf{EqDiam}(u', u) \wedge$$
$$\mathsf{EqDiam}(v', v) \wedge \mathsf{CONJ}(u, v, u', v'))) \wedge$$
$$\forall u,v((\mathsf{MCP}(u, y) \wedge \mathsf{MCP}(v, y) \wedge \neg u = v)$$
$$\rightarrow \exists u',v'(\mathsf{MCP}(u', x) \wedge \mathsf{MCP}(v', x) \wedge \mathsf{EqDiam}(u', u) \wedge$$
$$\mathsf{EqDiam}(v', v) \wedge \mathsf{CONJ}(u, v, u', v')));$$
$$\mathsf{CG}(x, y) =_{def}$$
$$\forall z((\mathsf{S\text{-}SR}(z) \wedge \mathsf{P}(z, x)) \rightarrow \exists w(\mathsf{S\text{-}CGR}(w, z) \wedge \mathsf{P}(w, y)))$$
$$\wedge$$
$$\forall z((\mathsf{S\text{-}SR}(z) \wedge \mathsf{P}(z, y)) \rightarrow \exists w(\mathsf{S\text{-}CGR}(w, z) \wedge \mathsf{P}(w, x))).$$

T2 → T1 (assuming HP_1) We first introduce C from which one obtains P, +, SC as in ⁄3.2, and PO[(2)] *partial overlap*. Then we use the definition of sphere given in [Borgo *et al.*, 1996]:

$$\mathsf{C}(x, y) =_{def} \forall z \exists z'(\mathsf{CG}(z', z) \wedge \mathsf{O}(z', x) \wedge \mathsf{O}(z', y));$$
$$\mathsf{PO}(x, y) =_{def} \mathsf{O}(x, y) \wedge \neg\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$$
$$\mathsf{S}(x) =_{def} \mathsf{SR}(x) \wedge \forall y((\mathsf{CG}(x, y) \wedge \mathsf{PO}(x, y)) \rightarrow \mathsf{SR}(x - y)).$$

T1 → T3 This direction follows from the relations based on P and S, see ⁄3.2 for their interpretations:

$$\mathsf{CONJ}(x, y, x', y') =_{def} \exists s_x,s_y,s'_x,s'_y(\mathsf{SCG}(s_x, s_y, s'_x, s'_y) \wedge$$
$$\forall s_{xc},s_{yc},s'_{xc},s'_{yc}((\mathsf{CNC}(s_{xc}, s_x) \wedge \mathsf{CNC}(s_{yc}, s_y) \wedge$$
$$\mathsf{CNC}(s'_{xc}, s'_x) \wedge \mathsf{CNC}(s'_{yc}, s'_y)) \rightarrow$$
$$\mathsf{O}(s_{xc}, x) \wedge \mathsf{O}(s_{yc}, y) \wedge \mathsf{O}(s'_{xc}, x') \wedge \mathsf{O}(s'_{yc}, y'))).$$

T1 → T6 P and S suffice to define BTW, see ⁄3.2 for their interpretations:

$$\mathsf{Conv}(x) =_{def} \forall s,s',s''(\mathsf{P}(s+s', x) \wedge \mathsf{BTW}(s'', s, s')) \rightarrow \mathsf{O}(s'',x);$$
$$\mathsf{ConvH}(x, y) =_{def} \neg\exists z(\mathsf{Conv}(z) \wedge \mathsf{P}(y, z) \wedge \mathsf{PP}(z, x)).$$

The details of these characterizations and their proofs are quite interesting in themselves. However, it is not possible to dwell on this subject here. In the following section we direct our attention to some general aspects and remarks.

4.1 Analysis of the Results

The strongest result we obtain is that, for dimension $n \leq 3$ and under assumption HP2, the sets of primitives of each theory T1-T5 are equivalent in the interpretation above. Furthermore, these "translations" make explicit several conceptual connections. Regarding T6, C and ConvH do not allow one to construct any notion of distance and this theory is less expressive compared to the others. Note that we assume restriction $n \leq 3$ on the dimension of the space only when defining S using P and CG. Besides this case, all the results hold in any dimension.

The authors of [Bennett *et al.*, 2000b] propose another definition of S from P and CG and they try to provide semantic equivalence in the domain of open regular regions. However, this attempt fails since it does not rule out regions like Reuleaux polytopes[15].

At present we do not know whether HP_2 can be disposed of, as we needed it in our proof of T3 → T2. The same assumption is used in the direct proof of T1 → T2 using the definition proposed in [Borgo *et al.*, 1996].

Among the other direct links, T2 → T3 is a simple exercise, whereas a direct connection T3 → T1 is quite complex (see [Tech. Rep. 2001]). T3, T4, and T5 have simpler connections. Assumption HP_1 is necessary in defining C from Closer to impose that two closed regions at distance zero share a point. We conclude remarking that C and therefore P are definable from Conj (or Ccon) without assumption HP_1. As a result, primitive P is not necessary in T3.

4.2 Impact of the Choice of the Domain

We already pointed out in section ⁄3 that similar results hold when we compare these theories in the domain of closed regular regions. It is interesting to see how the hypotheses on the domain affect the proofs. In this way, one provides a rough estimation of the independence of the results from the selected context. For example, we know from [Randell and Cohn, 1992; Masolo and Vieu, 1999] that P cannot be defined from C (with the interpretations given in ⁄3.2) in an atomic theory. A similar result holds for C and SR in T2.

The definition of S in terms of P, SR, and CG falls short of capturing sphericity when some restriction on the isometries of the space is introduced: in a set of rectangles with only translations as isometries, any rectangle does satisfy the definition of sphere. Despite this dependence, the definition of S is cognitively relevant. A region *is* a sphere whenever isotropic[16] flaws cannot be detected.

The case of S is paradigmatic, though this situation holds for most of the definitions considered. In general, all our proofs depend on the properties of R^n. (Here connection T3 → T4 is the only exception.) These dependencies are not surprising and one has to consider these results in the given context. At the same time, it is clear that these definitions provide a major opportunity to clarify conceptual relationships and intended meanings.

5 Conclusions

We have developed a method for comparing mereogeometries and presented several new results about well-known theories. Such a comparison emphasizes the cognitive aspects of these theories and minimizes the degree of formal characterization with respect to standard logical techniques. It allows us to shed light on the relationships between formalization, expressive power and intuitive meaning. It also improves the understanding of each theory

[14] This holds even without assumption HP_1. As we have seen: (*i*) CCon can be defined by Conj without P and (*ii*) C can be defined using CCon. From this, the other relations follow and so P is dispensable.

[15] An example in R^2 is the region obtained by intersecting the three discs centered at the three vertices of an equilateral triangle and with radius equal to the side of the triangle.

[16] A region is isotropic if invariant with respect to directions.

highlighting properties of their primitives and relationships with respect to other theories. This provides a reliable base of comparison for the pratictioner.

However, our approach is not a substitute for usual logical methods. The results obtained with this method depend on the particular domain and interpretations selected. Further work is needed to fully understand the problematic aspects that arise in a comparison based on intuitive interpretations and to provide a satisfactory set of criteria for an optimal application of the method. Possible developments include: (*i*) widening the classes of theories that can be considered in a single comparison; (*ii*) controlling/modifying of the general setting provided in the framework R^n; and (*iii*) generalizing the method to other frameworks to allow different comparisons.

References

[Asher and Vieu, 1995] Nicholas Asher and Laure Vieu. Toward a Geometry of Common Sense: A Semantics and a Complete Axiomatization of Mereotopology. In *International Joint Conference on Artificial Intelligence (IJCAI95)*, pp. 846-852, Montreal, 1995.

[Aurnague *et al.*, 1997] Michel Aurnague, Laure Vieu and Andr e Borillo. La repr sentation formelle des concepts spatiaux dans la langue. In Michel Denis (ed.) *Langage et cognition spatiale*, pp. 69-102, Masson, Paris, 1997.

[Bennett *et al.*, 2000a] Brandon Bennett, Anthony G. Cohn, Paolo Torrini and Shyamanta M. Hazarika. Describing Rigid Body Motions in a Qualitative Theory of Spatial Regions. In *AAAI '00*, pp. 503-509, , 2000a.

[Bennett *et al.*, 2000b] Brandon Bennett, Anthony G. Cohn, Paolo Torrini and Shyamanta M. Hazarika. A Foundation for Region-Based Qualitative Geometry. In *ECAI '00*, pp. 204-208, , 2000b.

[Biacino and Gerla, 1991] Loredana Biacino and Giangiacomo Gerla. Connection Structures. *Notre Dame Journal of Formal Logic*, 32(2):242-247,1991.

[Borgo *et al.*, 1996] Stefano Borgo, Nicola Guarino and Claudio Masolo. A Pointless Theory of Space based on Strong Connection and Congruence. In *Principles of Knowledge Representation and Reasoning (KR96)*, pp. 220-229, Boston, MA, 1996.

[Casati and Varzi, 1999] Roberto Casati and Achille C. Varzi. *Parts and Places.* MIT Press, Cambridge, 1999.

[Clarke, 1985] B. Clarke. Individuals and Points. *Notre Dame Journal of Formal Logic*, 26(1):61-75,1985.

[Clarke, 1981] B. L. Clarke. A Calculus of Individuals Based on "Connection". *Notre Dame Journal of Formal Logic*, 22:204-18,1981.

[Cohn and Varzi, 1998] Anthony Cohn and Achille Varzi. Connection Relations in Mereotopology. In *ECAI '98*, pp. 150-154, Wiley, 1998.

[Cohn, 1995] Anthony G. Cohn. Qualitative Shape Representation using Connection and Convex Hulls. In *Time, Space and Movement: Meaning and Knowledge in the Sensible World*, pp. 3-16 (part C), Toulouse, 1995.

[de Laguna, 1922] Theodore de Laguna. Point, Line and Surface, as Sets of Solids. *The Journal of Philosophy*, XIX(17):449-461,1922.

[Dugat *et al.*, 1999] Vincent Dugat, Pierre Gambarotto and Yannick Larvor. Qualitative Theory of Shape and Orientation. In *IJCAI'99 Workshop: Hot Topics in Spatial and Temporal Reasoning*, pp. 45-53, Arlington, USA, 1999.

[Gerla, 1994] Giangiacomo Gerla. Pointless Geometries. In F. Buekenhout (ed.) *Handbook of Incidence Geometry*, pp. , Elsevier Science Publishers, , 1994.

[Grzergorczyk, 1960] A. Grzergorczyk. Axiomatizability of Geometry Without Points. *Synthese*, 12:109-127,1960.

[Masolo and Vieu, 1999] Claudio Masolo and Laure Vieu. Atomicity vs. Infinite Divisibility of Space. In *COSIT'99*, pp. 235-250, Stade, Germany, 1999.

[Nicod, 1962] Jean Nicod. *La geometrie dans le monde sensible.* Presses Universitaires de France, Paris, 1962.

[Pratt and Lemon, 1997] Ian Pratt and Oliver Lemon. Ontologies for Plane, Polygonal Mereotopology. *Notre Dame Journal of Formal Logic*, 38(2):225-245,1997.

[Randell and Cohn, 1992] D. A. Randell and A. G. Cohn. A Spatial Logic Based on Regions and Connections. In B. Nebel, C. Rich and W. Swartout (eds.), *Principles of Knowledge representation and Reasoning. Proceedings of the Third International Conference*, pp. 165-176, Morgan Kaufmann, Los Altos, 1992.

[Simons, 1987] P. Simons. *Parts: a Study in Ontology.* Clarendon Press, Oxford, 1987.

[Stock, 1997] Oliviero Stock, editor. *Spatial and Temporal Reasoning*. Kluwer Academic, Dordrecht, 1997.

[Tarski, 1956] Alfred Tarski. Foundations of the geometry of solids. In J. Corcoran (ed.) *Logic, semantics, metamathematics*, pp. 24-30, Oxford University Press, Oxford, 1956.

[Whitehead, 1929] Alfred N. Whitehead. *Process and Reality.* Macmillan, New York, 1929.