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FULL MEREOGEOMETRIES

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Abstract. We analyze and compare geometrical theories based on mereology (*mereogeometries*). Most theories in this area lack in formalization, and this prevents any systematic logical analysis. To overcome this problem, we concentrate on specific interpretations for the primitives and use them to isolate comparable models for each theory. Relying on the chosen interpretations, we introduce the notion of environment structure, that is, a minimal structure that contains a (sub)structure for each theory. In particular, in the case of mereogeometries, the domain of an environment structure is composed of particular subsets of R^n. The comparison of mereogeometrical theories within these environment structures shows dependencies among primitives and provides (relative) definitional equivalences. With one exception, we show that all the theories considered are equivalent in these environment structures.

1. Introduction. At the time Lobachevskii (1835) published "New principles of geometry with complete theory of parallels," the axiomatic foundation of geometry was based on points. Such a formal system, called *Euclidean geometry*, falls short of satisfying cognitive concerns since it aims at modeling physical space relying on the abstract notion of point. The matter in dispute is that human experience of space is experience in magnitude and points cannot be empirically experienced. This simple observation makes evident the need for cognitively and philosophically sound geometrical systems whose formal study began in the 19th century (although it has received less emphasis with respect to the contemporary and orthogonal research on the fifth Euclidean axiom).

Taking *solids* as basic entities in his system, Lobachevskii revolutionizes the foundations of geometry from the ontological viewpoint and shows how to fill the gap between geometrical and spatial entities. As it happens often with revolutionary approaches, the work of Lobachevskii is quite obscure, and it is presented only informally. One has to wait almost a century to find a formal presentation of the new approach.

The theories developed by Whitehead (1929) (see also Biacino & Gerla, 1991, 1996, for a formal characterization of the theory of Whitehead), De Laguna (1922), Nicod (1924), Tarski (1956a), and Grzegorczyk (1960) aim at showing that the concept of point is not necessary in the foundation of geometry and, consequently, that the conceptualization of space can differ on several aspects: properties of the space (e.g., Euclidean vs. non-Euclidean geometries), primitive relations (e.g., being aligned, equidistance), and ontological nature of entities.[1] From the viewpoint of geometrical construction, these approaches

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[1] Instead of talking of *solids*, these authors refer to *regions*, *bodies*, or *volumes*. In some cases, these notions are not deeply characterized, thus it is difficult to understand if they presuppose a different intuition about physical objects and their possible locations in space. In this paper, *entity* and *region* are taken as generic and intuitive notions. In addition, some authors have recently developed theories based on domains containing entities of different dimensions, e.g., points, lines, and surfaces. See Gotts (1996) and Galton (1996, 2004).

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revolutionize the classical method of defining regions as sets of points. Indeed, here points are treated as particular sets of regions.[2] Since the new theories succeed in defining Euclidean entities and relations within a different logical domain, one cannot rely on purely formal arguments to establish which entities and relations deserve the role of geometrical primitives. Euclidean geometry is now challenged at the level of the basic entities, and external considerations start leaking into the geometry paradise.

The new geometries are justified by questions that arise outside the geometrical formalism itself and provide formal theories *adequate* for different tasks. In particular, region-based geometries seem to be cognitively more sound than point-based geometries since they make possible a direct mapping from empirical entities and laws to theoretical entities and formulas. Several issues need to be considered: the consequences of choosing extended regions as primitive entities, the meaning of experiencing empirically extended regions, and the doubts about perfect regions.[3] Following De Laguna, one wonders what it means to consider points to be sets of solids. Does it follow that the concept of point is defined in empirical terms? Idealized regions seem closer to empirical experience than Euclidean points; still they already require a form of abstraction. Once we admit this, it is not clear where to stop. Then, why should we reject the usual notion of point? De Laguna is aware of this problem: “Although we perceive solids, we perceive no abstractive set of solids (. . .) In accepting the abstractive set, we are as veritably going beyond experience as in accepting the solid of zero-length” (De Laguna, 1922, p. 460).

Beginning with the work of Clarke (1981, 1985), theories based on extended entities have attracted much interest for both their formal aspects[4] and their applicative potentialities. The ontological clearness and the evident connection with physical entities justify the philosophical interest in these theories. This approach receives particular emphasis in the field of *formal ontology*. Here one assumes the relations of *parthood* and *connection* to be basic notions that are exemplified by spatial or material entities like physical objects, chunks of matter, holes, etc. (see Simons, 1987; Casati & Varzi, 1999; Smith, 1998). Nowadays, one refers to these theories as *mereotopologies* since they are characterized by the combination of mereology (based on parthood) and topology (based on connection). Following this terminology, we call *mereogeometries* the theories that aim to reconstruct geometry extending mereotopological systems.

Mereogeometries are used in various areas. In Schmidt (1979), *physics* is presented as a theory based on extended entities. This theory allows to refer explicitly to the objects involved in experiments. Generally speaking, *cognitive science* and *computational linguistics* analyze the possibility of formalizing human learning, conceptualization, and categorization of spatial entities and relations. In particular, Knauff *et al.* (1997) and Renz *et al.* (2000) take into account the cognitive adequacy of topological relations, while Aurnague *et al.* (1997) and Muller (1998a) show how mereogeometrical notions are central in the semantics of natural language. Donnelly (2001) formalizes the theory of De Laguna in the perspective of *common-sense analysis* of spatial concepts. In *computer science* and

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[2] In particular, points are often defined as filters of regions. For a detailed discussion on the construction of points in mereotopology, see Biacino & Gerla (1991).

[3] Tarski defines points as classes of concentric spheres. However, in nature we do not find perfectly spherical objects. One can argue similarly about fractal-shaped regions or regions with an infinitely oscillating boundary (see Pratt-Hartmann & Schoop, 2000, for a discussion on ‘pathological’ regular regions).

[4] See Gerla (1994) for a good survey of mathematical research in this area. An analysis in terms of lattices is given in Stell (2000) and in terms of algebras in Düntsch *et al.* (2001).

more specifically in *qualitative spatial representation and reasoning* (see Cohn & Hazarika, 2001; Vieu, 1997, for good overviews), mereogeometries are applied for modeling qualitative morphology and movement of physical bodies (Bennett, 2001; Bennett *et al.*, 2000a,b; Borgo *et al.*, 1996; Cristani *et al.*, 2000; Dugat *et al.*, 1999; Muller, 1998b; Galton, 2000; Li & Ying, 2003; Randell & Cohn, 1989, 1992), for describing geographical spaces and entities in *Geographical Information Systems* (Pratt-Hartmann & Lemon, 1997; Pratt-Hartmann & Schoop, 2000; Stock, 1997), as well as for characterizing *medical* and *biological information* (Schulz & Hahn, 2001; Cohn, 2001; Smith & Varzi, 1999; Donnelly, 2004).

In all these areas, specific foundational and applicative concerns affect the development of the theories based on geometrically extended entities. Indeed, in the literature, there are numerous mereogeometries that differ on primitive entities, formal properties, as well as general principles. One surmises that this variety of systems has motivated a plethora of results on their relative strengths and drawbacks. Surprisingly, reading the literature, one cannot find an extended study of the relationships among these systems. Although extensive discussions on the cognitive, linguistic, and philosophical motivations for a theory are often undertaken, these are not accompanied by more formal considerations. The few arguments brought forward to discuss the relative expressive power are limited to antecedent versions of the presented theory and cannot be generalized to broader classes of systems. Such a lack of comparative analysis has practical reasons, in particular the poor axiomatization of most mereogeometries.

This being the situation, in the following sections, we try to fill the gap by presenting a method to systematically compare classes of mereogeometries. In our intentions, this method allows for a comparison of formal theories while concentrating on the meaning of the primitives. We consider it as a first answer to the need of assessment in this area. One can use this approach to state equivalences and similarities among the theories, in this way facilitating both reuse and communication among different applications. Since this task cannot be undertaken with the standard logical machinery (as mentioned above, most of the mereogeometries available in literature are only weakly formalized), the method we propose gives prominence to conceptual and ontological issues, issues that are at the center of the systems we are interested in. The goal is to make explicit important differences like description completeness and conceptual incompatibilities. Beside the direct advantage provided by a reliable classification method, such a comparison would help in selecting theories (perhaps according to the applicative or theoretical tasks one is facing), developing new theories, and extending or modifying those already available.

2. Conceptual comparison. Generally speaking, logical theories are compared at the syntactic or the semantic level. In the first case, one focuses on the interdefinability of primitive relations assumed in the theories to prove the equivalence of their axiomatics, while in the second case, one compares (classes of) models of these theories. Systematic analyses of these kinds have been developed on mereotopologies: Casati & Varzi (1999), Simons (1987), and Masolo & Vieu (1999) consider the syntactic level, while Biacino & Gerla (1991, 1996), Asher & Vieu (1995), Pratt-Hartmann & Schoop (1998, 2002), Roeper (1997), and Stell (2000) focus on the semantic level.

These kinds of comparison make sense if the theories are well characterized, i.e., the given axiomatization captures the *intended models* and so the *intended meanings* of the primitives. In the case of mereogeometries, only 2 theories have been proved to be semantically complete with respect to the models expressed in terms of R^n. Donnelly (2001) provided a full axiomatization for the theory of De Laguna (1922), which is based on the primitive *can connect*, whereas the theory of Tarski (1956a), based on the primitives *parthood* and *being a sphere*, is fully axiomatized by Bennett (2001).[5] The other mereogeometries available in the literature are only 'partially' axiomatized (they are not completely characterized with respect to the intended models), see Aurnague *et al.* (1997), Borgo *et al.* (1996), and Cohn (1995), or are axiomatized only indirectly relying on point-based axioms (Nicod, 1924).[6]

In order to overcome this lack of explicit or direct formalization and to carry out an exhaustive and informative analysis of the links between the different theories, we follow the approach delineated for the mereotopologies by Cohn & Varzi (2003) and compare the mereogeometries on the basis of their *intended models*. Cohn and Varzi take classical topology as a unifying framework for the comparison. In the case of mereogeometries, we rely on R^n since this system is generally used by the authors to describe the (intended) models of their theories.[7]

Some authors describe the intended models in a formal way, while in other cases, the models are only sketched. Therefore, our first task is to isolate interpretations in R^n that conform with the formal and informal descriptions and that are compatible with the given axiomatizations. We call any such interpretation a *natural interpretation*, and the underlying models are dubbed *natural models* of the theory. These notions are discussed below.

Our second (and main) task is to compare these natural/intended models within the chosen unifying framework, i.e., R^n. The analysis of the models (see The Theories and Their Interpretations section) reveals that they differ significantly on the primitive predicates adopted while the domains of interpretation, henceforth called *natural domains*, are quite similar. Indeed, all these domains are contained in the class of nonempty *regular regions*.[8] As a consequence, most of our work concentrates on the relationship among primitives. Technically speaking, we will proceed as follows: first, we collect all the primitives, say $P_1, \ldots, P_n$, in the systems we want to compare. Then, for each primitive P_i, we fix an interpretation R_i in the class D of regular regions of R^m (for some fixed m). Keeping the R_i's fixed, we define several *environment structures* $\langle D_j, R_1^j, \ldots, R_n^j\rangle$, where R_i^j is the restriction of R_i to $D_j \subseteq D$. We write Φ without indices for the most inclusive environment structure, namely $\Phi = \langle D, R_1, \ldots, R_n\rangle$.

[5] Tarski himself axiomatizes the primitives only indirectly. He first defines several relations among spheres (e.g., concentricity), relying on the intended interpretation of the primitives, and provides axioms only for parthood. Then, he introduces points as classes of concentric spheres. In this way, he can define equidistance among points using properties of concentric spheres and adopt the Euclidean axioms to constrain equidistance and, indirectly, the predicate *being a sphere*.

[6] Nicod considers the primitives *parthood* and *conjugation* (from which he defines points and their standard relationships) and assumes all theorems of the point-based Euclidean geometry as axioms to force the desired interpretation for the 2 primitives. He does not provide a direct set of axioms for the chosen primitives. Nicod is aware of the formal drawbacks of this approach. His main goal was to show that extended regions can be taken as the fundamental entities of geometry, and the method he applied does the job. As a result, the system has no proper axiomatization.

[7] This is not in contrast with the ontological nature of mereogeometries because R^n is used only as an 'environment' for intended models. Indeed, these rely on regions in R^n and not on single points. In addition, spatial theories adequate to cognitive or applicative tasks focus on qualitative relations and do not aim at capturing 'new' notions of space.

[8] A subset A of R^n is said to be a regular region if (a) the closure of A equals the set obtained by the topological closure of the biggest open set in A and (b) the interior of A equals the biggest open set contained in the topological closure of A itself, see Basic Notions in R^n section.

Given an environment structure $\Phi_j = \langle D_j, R_1^j, \ldots, R_n^j \rangle$ and a mereogeometry **T**, we define the *local structure* $\Phi_j(\mathbf{T})$ to be the structure obtained from Φ_j by dropping the sets R_i^j which are not the semantic counterpart of primitives in **T**, i.e., $\Phi_j(\mathbf{T})$ is the 'projection' of Φ_j on the primitives of **T**. For example, if P_1 and P_3 are the only primitives of **T**, then $\Phi_j(\mathbf{T}) = \langle D_j, R_1^j, R_3^j \rangle$. Similarly, we write $\Phi_j(\mathbf{T}+\mathbf{T}')$ for the structure obtained from Φ_j by dropping the sets R_i^j which are the semantic counterpart of primitives neither in **T** nor in **T**′, i.e., the projection of Φ_j on the union of primitives of **T** and **T**′. These local structures furnish the backbone of our comparison. As a consequence, our strategy is structure dependent and constitutes a generalization of more traditional comparison methods.

Using environment structures, one can formalize the notion of *conceptually equivalent theories*. This is the motivation for the definitions below. Let **T**, **T**′ be 2 mereogeometries.

Definition 1. *If P is a primitive of* **T***, we say that P is* explicitly Φ_j-definable *in* **T**′ *if there exists an expression φ in the language of* **T**′ *such that the interpretations of P and φ are equivalent in the local structure $\Phi_j(\mathbf{T} + \mathbf{T}')$. Expression φ is called a Φ_j-definition of P in* **T**′.

Definition 2. *A theory* **T** *is a* Φ_j-subtheory *of* **T**′ *if every primitive P of* **T** *has an explicit Φ_j-definition in* **T**′.

Definition 3. *Two theories* **T** *and* **T**′ *are* Φ_j-equivalent *if* **T** *is a Φ_j-subtheory of* **T**′ *and* **T**′ *is a Φ_j-subtheory of* **T**.

Definition 4. *Let* **T** *and* **T**′ *be theories with natural domains D_i and D_j, respectively. We say that* **T** *and* **T**′ *are* conceptually equivalent *if they are both Φ_i-equivalent and Φ_j-equivalent.*

The notions of Φ_j-equivalent and conceptual equivalence call attention to the domains of interpretation and to the expressive power of the systems. In our terminology, 2 theories are Φ_j*-equivalent* if, roughly speaking, when interpreted in the domain D_j their primitives have the same expressive power. Now, assume that we have a first-order translation between 2 Φ_j-equivalent theories. Given a deductively complete axiomatization of the first theory in D_j, this furnishes a complete axiomatization of the second theory as well (of course, such an axiomatization is relative to the given domain D_j).

In the case where **T** has natural domain D_i and **T**′ has natural domain D_j, the fact that they are Φ_i-equivalent and Φ_j-equivalent tells us that **T** is a true *conceptual counterpart* of **T**′ (and vice versa) since one theory captures the natural model(s) of the other when it is interpreted over the corresponding domain. Finally, note that the notion of conceptual equivalence is independent from the overall set of theories one is considering, that is, from the overall environment structure. Indeed, the inclusion (or exclusion) of other theories does not alter the results about **T** and **T**′.

Some mereogeometries already furnish definitions that aim to capture primitives of other theories. In these cases, it is crucial to verify whether the defined relations really correspond to those primitives. For example, the theory in Donnelly (2001) (theory **T4** of Mereogeometries section), defines the relation *connection* (C) in terms of the primitive *can connect* (CCon) as follows:

$$\mathsf{C}(x, y) =_{\text{def}} \forall z(\mathsf{CCon}(z, x, y)). \tag{2..1}$$

Since C is a primitive in the theory of Cohn (1995) (theory **T6** of Mereogeometries section), its interpretation must agree with the interpretation obtained by (2.1). In this case, one must verify that the interpretation of C defined as in (2.1), which depends on the interpretation of CCon, conforms with the natural interpretation of C given by Cohn (1995) in the domains associated to these theories. A crucial step in our comparison is to provide this kind of analysis.

We hasten to point out that this method is not universal and not always straightforward. Sometimes, it is hard to isolate a meaningful environment for comparison or it might turn out that a complete comparative analysis is too complex to be carried out. Some issues based on these considerations are discussed in Environment Structures section. Also, it is important to take into account that existential axioms (taken to constrain the domain of interpretation for a given theory) might fail in environment structures with restricted domains.

In the next section, we give a description of the mereogeometries studied in this paper together with their natural models expressed in R^n. In Environment Structures section, we fix and justify our choice of environment structures, and in Translations Between Theories section, we present the details of the comparison verifying the explicit syntactic translations across the theories and introducing new or corrected translations whenever necessary.

3. Mereogeometries. In this section, we present the mereological systems considered in this paper and fix their formal interpretations. Since R^n is the common underlying framework, we begin by listing some standard relations of this system and then use them to interpret the mereogeometrical vocabulary.

3.1. Basic notions in $\mathbf{R}^n$. Here we recall some topological and geometrical relations and functions on R^n; these are needed for interpreting mereogeometry:

- topological operators: *closure* ([]), *interior* ($^\circ$);
- Euclidean distance, dist: $\mathrm{R}^n \times \mathrm{R}^n \to [0, +\infty)$;
- standard operators, functions, and relations definable from these.

Also, recall that a subset A of R^n is said to be a *regular region* whenever $[A]^\circ = A^\circ$ and $[A^\circ] = [A]$.

In the following list of operators and relations, lowercase variables stand for points of R^n and uppercase variables for regular regions in R^n.

Operators and functions in R^n (X and Y nonempty):

$\mathrm{ball}(c, r) = \{x | \mathrm{dist}(x, c) < r\}$, where $r > 0$; (nonempty open ball of radius r, center c)
$\partial(X) = [X] - X^\circ$; (boundary of X)
$\mathrm{diam}(X) = \sup\{\mathrm{dist}(x, y) | x, y \in X\}$; (diameter of X)
$\mathrm{dist}(X, Y) = \inf\{\mathrm{dist}(x, y) | x \in X \wedge y \in Y\}$;[9] (distance between X and Y)

Relations R^n (X and Y nonempty):

$\mathrm{Btw}(x, y, z)$ iff $\mathrm{dist}(y, x) + \mathrm{dist}(x, z) = \mathrm{dist}(y, z)$; ($x$ is between y and z)
$\mathrm{Congr}(X, Y)$ iff there exists an isometry f such that
$f(X) = Y$; (X is congruent to Y)

9 Sometimes, we write $\mathrm{dist}(x, Y)$ for $\mathrm{dist}(\{x\}, Y)$ and analogously for $\mathrm{dist}(X, y)$. Also, note that we use the logical symbols $\wedge, \forall, \to$, etc. both in the mereogeometrical languages and in the semantic statements.

$Conv(X)$ iff $\forall x,y,z((x,y \in X \wedge$
$\mathrm{Btw}(z, x, y)) \rightarrow z \in X)$; ($X$ is a convex region)
$\mathrm{Conx}(X)$ iff $\forall A,B((A^\circ \neq \emptyset \wedge$
$B^\circ \neq \emptyset \wedge X^\circ = A^\circ \cup B^\circ) \rightarrow A^\circ \cap B^\circ \neq \emptyset)$; ($X$ is a connected region)
$\mathrm{WConx}(X)$ iff $\forall A,B((A^\circ \neq \emptyset \wedge$
$B^\circ \neq \emptyset \wedge X^\circ = A^\circ \cup B^\circ) \rightarrow [A] \cap [B] \neq \emptyset)$; ($X$ is a weakly connected region)
$\mathrm{WWConx}(X)$ iff $\forall A,B((A^\circ \neq \emptyset \wedge B^\circ \neq$
$\emptyset \wedge X^\circ = A^\circ \cup B^\circ) \rightarrow \mathrm{dist}(A, B) = 0)$; ($X$ is a ω-weakly connected region).

From the definitions, for all $X \subseteq \mathrm{R}^n$, we have $\mathrm{Conx}(X) \rightarrow \mathrm{WConx}(X) \rightarrow \mathrm{WWConx}(X)$. The converse does not hold. However, we have $(\mathrm{WWConx}(X) \wedge \mathrm{diam}(X) < +\infty) \rightarrow \mathrm{WConx}(X)$. These results are proven in Appendix A. Other topological and geometrical lemmas based on the notions given above become handy in proving theorems of later sections. These lemmas are collected in Appendix A as well.

3.2. The theories and their (natural) interpretations. As we have seen, the natural interpretation of the nonlogical primitives is crucial for the comparison. Because of this, we provide detailed notes with references to the literature and point out the cases where the information available is not satisfactory.

In this section, we list the nonlogical vocabulary V_j, the domain D_j, and the (natural) interpretation $[\![\cdot]\!]_\mathrm{j}$ of each mereogeometry **Tj** we consider. Also, assume that an assignment function $\Im_n$ from the set of variables to regular regions in R^n has been fixed for each index n. If $D \subseteq \wp(\mathrm{R}^n)$, then $[\![\mathsf{R}(x, y)]\!]_j$ and $[\![\mathsf{R}]\!]_j(X, Y)$ stand for $[\![\mathsf{R}]\!]_\mathrm{j}(\Im_n(x), \Im_n(y))$, where $\Im_n(x) = X$ $(X \in D_j \subseteq D)$. Finally, since there is no danger of confusion, throughout the paper we write $[\![x]\!]_j$ for $\Im_n(x)$ whenever the index n is fixed by the context.

T1 – Theory presented in Tarski (1956a) and further developed in Bennett (2001) and Bennett *et al.* (2000b):
$V_1 = \{\mathsf{P}, \mathsf{S}\}$, where
$\mathsf{P}(x, y)$ stands for 'x is part of y' and
$\mathsf{S}(x)$ for 'x is a sphere';
D_1 = {nonempty regular open subsets of R^n}
$= \{X \subseteq \mathrm{R}^n | X \neq \emptyset \wedge [X]^\circ = X\}$;
$[\![\mathsf{P}(x, y)]\!]_1 = X \subseteq Y$;
$[\![\mathsf{S}(x)]\!]_1 = \exists c \in \mathrm{R}^n, r \in \mathrm{R}^+ \ (X = \mathrm{ball}(c, r))$.

T2 – Theory presented in Borgo *et al.* (1996):[10]
$V_2 = \{\mathsf{P}, \mathsf{SR}, \mathsf{CG}\}$, where
$\mathsf{P}(x, y)$ stands for 'x is part of y',
$\mathsf{SR}(x)$ for 'x is a simple region' (or 'x is connected'), and
$\mathsf{CG}(x, y)$ for 'x is congruent to y';
D_2 = {nonempty regular open subsets of R^n with finite diameter}
$= \{X \subseteq \mathrm{R}^n | X \neq \emptyset \wedge [X]^\circ = X \wedge \mathrm{diam}(X) < +\infty\}$;
$[\![\mathsf{P}(x, y)]\!]_2 = X \subseteq Y$;
$[\![\mathsf{SR}(x)]\!]_2 = \mathrm{Conx}(X)$;
$[\![\mathsf{CG}(x, y)]_2 = \mathrm{Congr}(X, Y)$.

[10] This theory is restricted to R^3 since it was developed for the description of physical objects. One can easily generalize it to R^n by varying the axioms on the space dimension.

T3 – Theory given in Nicod (1924):[11]
V_3 = {P, Conj}, where
P(x, y) stands for 'x is part of y' and
Conj(x, y, x', y') for 'x,y and x',y' are conjugates';
D_3 = {nonempty regular closed connected subsets of R^n}
$= \{X \subseteq \mathrm{R}^n | X \neq \emptyset \wedge [X^{\circ}] = X \wedge \mathrm{Conx}(X)\}$;
$[\![\mathsf{P}(x, y)]\!]_3 = X \subseteq Y$;
$[\![\mathsf{Conj}(x, y, x', y')]\!]_3 = \exists p, q, p', q'(p \in X \wedge q \in Y \wedge p' \in X' \wedge q' \in Y' \wedge \mathrm{dist}(p, q)) = \mathrm{dist}(p', q'))$.

T4 – Theory introduced in De Laguna (1922) and further developed in Donnelly (2001):[12]
V_4 = {CCon}, where
CCon(x, y, z) stands for 'x can connect both y and z';
D_4 = {nonempty regular closed connected subset of R^n with finite diameter}
$= \{X \subseteq \mathrm{R}^n | X \neq \emptyset \wedge [X^{\circ}] = X \wedge \mathrm{Conx}(X) \wedge \mathrm{diam}(X) < +\infty\}$;
$[\![\mathsf{CCon}(x, y, z)]\!]_4 = \mathrm{dist}(Y, Z) \leq \mathrm{diam}(X)$.

T5 – Theory first introduced in Van Benthem (1983) and further developed in Aurnague *et al.* 1997:[13]
V_5 = {C, Closer}, where
C(x, y) stands for 'x is connected to y' and
Closer(x, y, z) for 'x is closer to y than to z';
D_5 = {nonempty regular subsets of R^n}[14]
$= \{X \subseteq \mathrm{R}^n | X \neq \emptyset \wedge ([X] = [X^{\circ}] \wedge X^{\circ} = [X]^{\circ})\}$;
$[\![\mathsf{C}(x, y)]\!]_5 = X \cap Y \neq \emptyset$;
$[\![\mathsf{Closer}(x, y, z)]\!]_6 = \mathrm{dist}(X, Y) < \mathrm{dist}(X, Z)$.

T6 – Theory given in Cohn (1995) and Cohn *et al.* (1997a,b):
V_6 = {C, ConvH}, where
C(x, y) stands for 'x is connected to y' and
ConvH(x, y) for 'x is the convex hull of y';
D_6 = {nonempty regular open subsets of R^n}
$= \{X \subseteq \mathrm{R}^n | X \neq \emptyset \wedge [X]^{\circ} = X\}$;

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[11] Nicod provides an informal description of Conj: "[D]eux volumes AA', BB' sont *conjugués* s'il existe un point de A et un point de A', un point de B et un point de B', séparés par la même distance" (Nicod, 1924, pp. 27–28). He also characterizes the domain, see (Nicod, 1924, p. 27).

[12] De Laguna provides an informal description of CCon: "[T]o say that C can connect A and B would be understood to mean that we could, if we wished, put C in simultaneous contact with A and B" (De Laguna, 1922, p. 450), i.e., considering a domain of closed regions, $[\![\mathsf{CCon}(x,y, z)]\!] = \exists p, p', q, r(p, p' \in X \wedge q \in Y \wedge r \in Z \wedge \mathrm{dist}(p, p') = \mathrm{dist}(q, r))$. In Donnelly (2001), a different interpretation is given: $[\![\mathsf{CCon}(x, y, z)]\!]_4 = \mathrm{dist}(Y, Z) \leq \mathrm{diam}(X)$. This interpretation is equivalent to the one of De Laguna in D_4, but, as shown in Definitions in **T1** section, in the domains containing nonconnected regions, it does not satisfy all the axioms provided by De Laguna. Since we are working in domains containing nonconnected regions as well, we consider the weaker one.

[13] No intended interpretation for the relation Closer is provided. The interpretation we propose seems self-evident and satisfies all the axioms. As far as we can tell, it is faithful to this approach.

[14] Van Benthem (1983) considers only convex and bounded regions.

$[\![\mathsf{C}(x, y)]\!]_6 = [X] \cap [Y] \neq \emptyset$;
$[\![\mathsf{ConvH}(x, y)]\!]_6 = Conv(X) \wedge Y \subseteq X \wedge \neg\exists Z(Conv(Z) \wedge Y \subseteq Z \wedge Z \subset X)$.

4. Environment structures. Having listed all the systems and their (natural) interpretations, the next issue is the definition of the environment structures for the comparison. The key step in this part of our method is the choice of the structure domains.

As we have seen, all mereogeometries in the Mereogeometries section consider nonempty regular regions and do not refer to lower dimension objects like points or boundaries. Looking at the descriptions, we see that all the domains, with the only exception of D_5, contain regular regions only, and these are either all open or all closed (perhaps with additional constraints). In particular, $D_1 = D_6 = \{X \subseteq \mathrm{R}^n | X \neq \emptyset \wedge X = [X]^\circ\}$ is the set of regular open regions. Let us call this domain D_O. From the topological point of view, D_O is the counterpart of $D_C = \{X \subseteq \mathrm{R}^n | X \neq \emptyset \wedge X = [X^\circ]\}$, the set of regular closed regions, since there exists a bijection $f\colon D_\mathrm{O} \to D_\mathrm{C}$ given by $f(X) = [X]$ such that $f^{-1}(X) = X^\circ$, $f^{-1}(f(X)) = X$, and $f(f^{-1}(X)) = X$.

Domain D_5 is the union of D_O and D_C. In D_5, set inclusion ($\subseteq$) is nonextensional: if X is a regular closed region in D_5, then $X^\circ \in D_5$ and $X^\circ \subset X$, but there is nothing in D_5 that makes the difference between these regions. This is a major difference between **T5** and the other systems, and it has far-reaching consequences already at the mereotopological level (Cohn & Varzi, 2003). In this paper, we concentrate on $\subseteq$-extensional systems; thus, we leave out the domain D_5. Indeed, set inclusion is extensional in all the other theories of Mereogeometries section, and it is used extensively to prove the results of Translations Between Theories section. This does not mean that we dismiss **T5** altogether. Instead, we take the interpretations of C and Closer provided by the authors of **T5** and apply these in the other domains. Admittedly, this covers only one direction in the comparison of **T5** with the other systems, but the analysis of theories **T1–T6** in the natural domain of **T5** is quite complex and we leave this subject for future work.

Focusing on the relationship between D_O and D_C, one wonders if it is possible to reduce the comparison to domains containing open regions (or closed regions) only. Let us start with a specific example. Two of our theories are interpreted in the class of closed regions, namely **T3** (primitives P and Conj) and **T4** (primitive CCon). From Lemma L.5 (see Appendix A), we see that the interpretation of P, i.e., $\subseteq$, is independent from the open–closed distinction. The interpretation of CCon behaves analogously since from Lemmas L.8 and L.9, we have $\mathrm{dist}(X, Y) = \mathrm{dist}([X], [Y]) = \mathrm{dist}(X^\circ, Y^\circ)$ and $\mathrm{diam}(X) = \mathrm{diam}([X^\circ]) = \mathrm{diam}(X^\circ)$. The case of Conj is different. The natural interpretation (in D_C) is given by the formula

$$\exists p, q, p', q'(p \in X \wedge q \in Y \wedge p' \in X' \wedge q' \in Y' \wedge \mathrm{dist}(p, q) = \mathrm{dist}(p', q')). \quad (4.1)$$

If we want to make justice of this interpretation in the domain D_O, we should consider the topological closures of the variable values, that is, the following formula:[15]

$$\exists p, q, p', q'(p \in [X] \wedge q \in [Y] \wedge p' \in [X'] \wedge q' \in [Y'] \wedge \mathrm{dist}(p, q) = \mathrm{dist}(p', q')). \quad (4.2)$$

However, formulas (4.1) and (4.2) are equivalent in D_C only. For a counterexample in D_O, let $n = 1$ and take $X = (0, 1)$, $Y = (4, 5)$, $X' = (1, 2)$, and $Y' = (3, 4)$. Thus, in both D_O

[15] Recall that, in a formula, the interpretation of the variables is restricted to the domain $\mathrm{D_O}$. However, the formula itself may refer to regions outside $\mathrm{D_O}$, which justifies our use of [ ] in the formula.

and D_C, formula (4.2) coincides with the (informal) interpretation provided by the authors, while (4.1) is specialized to D_C since it takes advantage of the properties of that domain. In particular, taking (4.2) to be the interpretation for Conj, our comparison can be carried out in both the domains D_O and D_C. This example suggests a way to reliably restrict our analysis to a subclass of regular regions, say D_O, by allowing us to 'transfer' the results to the other, D_C. Indeed, taking the closure of the values given by the assignment function, that is, substituting $[X]$ for any occurrence of X in the interpretation of a formula, we can interpret in D_O also those primitives that are defined on the domain of regular closed regions.

Finally, since some primitives seem to occur in several systems, we need to verify that the corresponding interpretations are compatible. If this is not the case, these have to be taken as distinct primitives. In our case, 2 primitives occur in several theories: P is included in **T1**, **T2**, and **T3**, and C is included in **T5** and **T6**. P has the same interpretation in all the theories; thus, we can identify the primitive P in all these theories. Regarding C, its interpretation differs in the 2 domains D_5 and D_6. Since the interpretation in **T5** is based on a domain that we do not consider in this paper, we have to evaluate the adequacy of the original interpretation in the actual domains we use for the comparison. In the domain of open regions, the interpretation of C given by **T5** reduces C to standard overlap, a relation that is weaker than any connection relation. This result seems in contrast with the goals of the authors. In the case of closed regions, the interpretation of C provided by **T5** coincides with that of **T6**. These observations suggest that the interpretation of $\mathsf{C}(x,y)$ as 'the intersection of the closure of regions X and Y is nonempty' is more reliable. Nonetheless, some arbitrariness seems to be involved in this choice. We overcome this criticism by providing a definition of C in terms of Closer (the only other primitive of **T5**) that is compatible with the chosen interpretation for C. That is, a deeper analysis shows that the relation C is definable through Closer in D_O or D_C (this definition is given in Dispensable Primitives section). Furthermore, from this result and Proposition 1, one can see that Closer can define C even if one decides to interpret it in the other way, i.e., as the standard overlap relation. That is, the real interpretation of C in the domains we consider is irrelevant since, as we show, this relation is dispensable (see Dispensable Primitives section).

Putting things together, we end up with 8 distinct environment structures: Φ_α–Φ_θ. These have fixed vocabulary $V = \{\mathsf{C}, \mathsf{CCon}, \mathsf{CG}, \mathsf{Closer}, \mathsf{Conj}, \mathsf{ConvH}, \mathsf{P}, \mathsf{S}, \mathsf{SR}\}$ and fixed interpretation functions (Table 1) but different domains (Table 2). Recall that each theory is associated with a specific domain, i.e., the domain of its natural model. As a consequence, with the exception of theory **T5**, we associate each theory with the structure isolated by its natural domain and the interpretation functions given at the end of Environment Structures section and call this the *natural environment structure* for that theory (Table 2).

Definition 5. *Let* **T** *be a theory and D the domain of its natural model. The natural environment structure of* **T** *is the environment structure among the* Φ_α–Φ_θ *that has domain D.*

From Conceptual Comparison section, given 2 theories and a domain, say **T1**, **T4** and D_α, the theories are said to be Φ_α-equivalent iff the relation $\mathrm{dist}(X, Y) \leq \mathrm{diam}(Z)$ can be defined in the structure $\Phi_\alpha(\mathbf{T1}) = \langle D_\alpha, X \subseteq Y, \exists c,r(X^\circ = \mathrm{ball}\ (c, r))$ with $r \in \mathrm{R}^+\rangle$, and the relations $X \subseteq Y$ and $\exists c,r(X^\circ = \mathrm{ball}(c, r))$ can be defined in the structure $\Phi_\alpha(\mathbf{T4}) = \langle D_\alpha, \mathrm{dist}(X, Y) \leq \mathrm{diam}(Z)\rangle$. In other terms, both the structures $\Phi_\alpha(\mathbf{T1})$ and $\Phi_\alpha(\mathbf{T4})$ can be *definitionally expanded* to the structure $\langle D_\alpha, \mathrm{dist}(X, Y) \leq \mathrm{diam}(Z), X \subseteq Y, \exists c,r(X^\circ = \mathrm{ball}(c, r))$ with $r \in \mathrm{R}^+\rangle$, see Hodges (1997).

Before moving to the next section, we add a couple of words to motivate our choice of CCon's interpretation. The CCon primitive is introduced in a domain of connected regions, and there are different ways to generalize it to the more comprehensive domain we consider in this paper. At first sight, when working in the domain of all open regular regions, one might want to impose that in $\mathsf{CCon}(x, y, z)$, variable x must range over connected regions only. This constraint would capture the intuition that whenever a region ‘can connect’ 2 given regions, then it ‘can connect’ any 2 regions that are at closer distance. However, this constraint is too strong with respect to the underlying intuition that in R^1 accepts that the region $X = (0,1) \cup (2,3)$ ‘can connect’ $Y = (10,12)$ and $Z = (13,15)$. From a broader perspective, the problem is to understand which properties of the primitive that are guaranteed by the peculiarity of the original domain should be explicitly enforced in the more general interpretation. Our approach in these cases is to adopt the interpretation that makes the primitive weaker. Such a choice allows us to better analyze the import of the primitive. For the sake of completeness, note that one could informally interpret $\mathsf{CCon}(x, y, z)$ as ‘there are 2 points of x whose distance is equal to the distance between a point of y and a point of z.’ This interpretation has been discharged for the simple reason that it would make CCon a subcase of Conj.

5. Translations between theories. In the previous sections, we have prepared the elements for the formal comparison. The mereogeometries that we consider have been presented in Mereogeometries section, and the environment structures have been chosen and motivated in Environment Structures section. Now, we enter into the actual comparison giving the formal results. In this section, we collect the theorems, while their proofs, which are sometimes long and involved, are presented in the appendices.

Since in the previous section we have shown how to reduce the comparison to structures with domain in D_{O} only, we abuse the above notation by using $\Phi_\alpha - \Phi_\delta$ as natural environment structures even for theories whose natural domain D is contained in D_{C}.

5.1. Verifying the given explicit definitions. First, we prove that the explicit definitions, provided by each theory and of interest for the comparison, are satisfied in the natural

Table 1. *Interpretation of the vocabulary on nonempty regular regions of R^n.*

$[\![\mathsf{C}(x, y)]\!] = [X] \cap [Y] \neq \emptyset$

$[\![\mathsf{CCon}(z, x, y)]\!] = \mathrm{dist}(X, Y) \leq \mathrm{diam}(Z)$

$[\![\mathsf{CG}(x, y)]\!] = \mathrm{Congr}(X, Y)$

$[\![\mathsf{Closer}(z, x, y)]\!] = \mathrm{dist}(Z, X) < \mathrm{dist}(Z, Y)$

$[\![\mathsf{Conj}(x, y, x', y')]\!] = \exists x,y,x',y'(x \in [X] \wedge y \in [Y] \wedge x' \in [X'] \wedge y' \in [Y'] \wedge \mathrm{dist}(x, y)$

$= \mathsf{C}ist(x', y'))$

$[\![\mathsf{ConvH}(x, y)]\!] = Conv(X) \wedge Y \subseteq X \wedge \neg\exists Z(Conv(Z) \wedge Y \subseteq Z \wedge Z \subset X)$

$[\![\mathsf{P}(x, y)]\!] = X \subseteq Y$

$[\![\mathsf{S}(x)]\!] = \exists c,r(X^\circ = \mathrm{ball}(c, r))$ with $r \in \mathrm{R}^+$

$[\![\mathsf{SR}(x)]\!] = \mathrm{Conx}(X)$

In this case, we indicate the interpretation function with the double brackets $[\![\ ]\!]$ without indices.

Table 2. *Structures and their domains.*

STRUCTURE	DOMAIN	DOMAIN DESCRIPTION Nonempty regular regions and ...	NATURAL ENVIRONMENT OF
Φ_α	$D_\alpha = D_O = \{X \subseteq \mathrm{R}^n \mid X \neq \emptyset \wedge [X]^\circ = X\}$	open	**T1**, **T6**
Φ_β	$D_\beta = \{X \in D_O \mid \mathrm{diam}(X) < +\infty\}$	open and finite	**T2**
Φ_γ	$D_\gamma = \{X \in D_O \mid \mathrm{Conx}(X)\}$	open and connected	
Φ_δ	$D_\delta = \{X \in D_O \mid \mathrm{Conx}(X) \wedge \mathrm{diam}(X) < +\infty\}$	open, finite, and connected	
Φ_ε	$D_\varepsilon = D_C = \{X \subseteq \mathrm{R}^n \mid X \neq \emptyset \wedge [X^\circ] = X\}$	closed	**T3**
Φ_ζ	$D_\zeta = \{X \in D_C \mid \mathrm{diam}(X) < +\infty\}$	closed and finite	
Φ_η	$D_\eta = \{X \in D_C \mid \mathrm{Conx}(X)\}$	closed and connected	
Φ_θ	$D_\theta = \{X \in D_C \mid \mathrm{Conx}(X) \wedge \mathrm{diam}(X) < +\infty\}$	closed, finite, and connected	**T4**

environment structure for that theory. We will see that there is one exception: the definition proposed in **T1** to capture the relation CCon yields an interpretation function that does not satisfy the axiomatization of the primitive CCon given in **T4**. We will show how to modify such a definition to capture the correct meaning of the primitive. Regarding **T5**, we do not take into account the definitions it provides because they are conceived for the domain D_5.

First, we consider the (derived) interpretations of those mereotopological notions that receive a common definition in the theories **T1**–**T6**. These notions are extensively used in the rest of the paper.

Proposition 1. *In all the structures $\Phi_{\alpha-\delta}$, the following holds:*

(DP) Let $\mathsf{P}^*(x,y) =_{\mathrm{def}} \forall w(\mathsf{C}(x, w) \rightarrow \mathsf{C}(y, w))$, then
$[\![\mathsf{P}^*(x, y)]\!]_{\alpha-\delta} = X \subseteq Y$;[16] (*x* is a part of *y*)

(DPP) Let $\mathsf{PP}(x,y) =_{\mathrm{def}} \mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$, then
$[\![\mathsf{PP}(x,y)]\!]_{\alpha-\delta} = X \subset Y$; (*x* is a proper part of *y*)

(DO) Let $\mathsf{O}(x,y) =_{\mathrm{def}} \exists z(\mathsf{P}(z, x) \wedge \mathsf{P}(z, y))$, then
$[\![\mathsf{O}(x,y)]\!]_{\alpha-\delta} = X \cap Y \neq \emptyset$; (*x* and *y* overlap)

(DPO) Let $\mathsf{PO}(x,y) =_{\mathrm{def}} \mathsf{O}(x, y) \wedge \neg \mathsf{P}(x, y) \wedge \neg \mathsf{P}(y, x)$, then
$[\![\mathsf{PO}(x, y)]\!]_{\alpha-\delta} = X \cap Y \neq \emptyset \wedge \neg X \subseteq Y \wedge \neg Y \subseteq X$; (*x* and *y* partially overlap)

[16] Note that the interpretation of P* is identical to the interpretation fixed for P (Table 1). For this reason, in the following we will identify P* with P.

(D+) Let $\mathsf{SUM}(z, x, y) =_{\text{def}} \forall w(\mathsf{O}(w, z) \leftrightarrow (\mathsf{O}(w, x) \vee \mathsf{O}(w, y)))$, then
$[\![\mathsf{SUM}(z, x, y)]\!]_{\alpha-\delta} = (Z = [X \cup Y]^{\circ})$; ($z$ is the sum of x and y)

(D−) Let $\mathsf{DIF}(z, x, y) =_{\text{def}} \forall w(\mathsf{P}(w, z) \leftrightarrow (\mathsf{P}(w, x) \wedge \neg\, \mathsf{O}(w, y)))$, then
$[\![\mathsf{DIF}(z, x, y)]\!]_{\alpha-\delta} = (Z = X - [Y])$; ($z$ is x minus y)

(DEC) Let $\mathsf{EC}(x, y) =_{\text{def}} \mathsf{C}(x, y) \wedge \neg\, \mathsf{O}(x,y)$, then
$[\![\mathsf{EC}(x, y)]\!]_{\alpha-\delta} = [X] \cap [Y] \neq \emptyset \wedge X \cap Y = \emptyset$. ($x$ and y are externally connected)

(DIP) Let $\mathsf{IP}(x,y) =_{\text{def}} \mathsf{P}(x, y) \wedge \forall z(\mathsf{C}(z, x) \rightarrow \mathsf{O}(z, y))$, then
$[\![\mathsf{IP}(x, y)]\!]_{\alpha-\delta} = [X] \subseteq Y$; ($x$ is an interior part of y)

(DTPP) Let $\mathsf{TPP}(x, y) =_{\text{def}} \mathsf{PP}(x, y) \wedge \exists z(\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y))$, then
$[\![\mathsf{TPP}(x,y)]\!]_{\alpha-\delta} = X \subset Y \wedge \partial(X) \cap \partial(Y) \neq \emptyset$; ($x$ is an tangential proper part of y)

(DSC) Let $\mathsf{SC}(x) =_{\text{def}} \forall y,z(\mathsf{SUM}(x, y, z) \rightarrow \mathsf{C}(y, z))$, then
$[\![\mathsf{SC}(x)]\!]_{\alpha-\delta} = \mathrm{WConx}(X)$; ($x$ is weakly connected).

Proof. See Appendix B.1. □

We are now ready to analyze the definitions provided in some theories, namely **T1**, **T2**, **T4**, and **T6**. The goal here is to ensure that these definitions capture correctly the intended notions, to provide counterexamples where they do not, and to propose a corrected version when needed. To distinguish the vocabulary of the environment structures in Environment Structures section from the relation symbols within a theory, we label those in the latter group with the index of the theory where they occur. For instance, $\mathsf{C}_1(x,y)$ is the connection relationship defined in theory **T1**, while $\mathsf{C}(x,y)$ is the connection relationship with interpretation as in Table 1.

5.1.1. Definitions in T1. *Primitives of* **T1**: P, S. The explicit definitions provided in this theory involve 2 relationships that we do not discuss directly. The first, CNC, is the relationship that holds between 2 concentric spheres and is introduced and defined by Tarski (1956a). More precisely, in this paper Tarski proves $[\![\mathsf{CNC}(x, y)]\!] = \exists c \in \mathrm{R}^n, r, r' \in \mathrm{R}^r (X = \mathrm{ball}(c, r) \wedge Y = \mathrm{ball}(c, r'))$. We do not repeat the argument and refer the reader to that paper on this topic. CNC is adopted in **T1** without changes, and it is used to define the connection relation as shown in definition (DC1) below. The other is CG, which is needed to capture CCon. The definition of CG given in the language of **T1** is quite complex. In From **T1** to **T2** section, we provide an improved definition of CG (within theory **T1**) that works in all the domains. There we also prove that its interpretation corresponds to that of Environment Structures section. For the time being, we show that the existing definition of CCon given in Bennett (2001) and Bennett *et al.* (2000b) must be corrected.

Explicit definitions furnished in **T1**:

(DC1) $\mathsf{C}_1(x, y) =_{\text{def}} \exists z(\mathsf{S}(z) \wedge \forall z'(\mathsf{CNC}(z', z) \rightarrow (\mathsf{O}(z', x) \wedge \mathsf{O}(z', y))))$;
(DSR1) $\mathsf{SR}_1(x) =_{\text{def}} \forall y, z(\mathsf{SUM}(x, y, z) \rightarrow \exists s(\mathsf{S}(s) \wedge \mathsf{O}(s, y) \wedge \mathsf{O}(s, z) \wedge \mathsf{P}(s, x)))$;
(DCC1) $\mathsf{CCon}_1(z, x, y) =_{\text{def}} \exists z'(\mathsf{CG}(z', z) \wedge \mathsf{C}_1(z', x) \wedge \mathsf{C}_1(z', y))$.

Proposition 2. $[\![\mathsf{C}_1(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{C}(x, y)]\!]$ *and* $[\![\mathsf{SR}_1(x)]\!]_{\alpha-\delta} = [\![\mathsf{SR}(x)]\!]$.

Proof. See Appendix B.2. □

Proposition 3. *Let* $[\![\mathsf{CG}(x,y)]\!] = Congr(X,Y)$*, then* $[\![\mathsf{CCon}_1(z,x,y)]\!]_\alpha = \exists z,z',x,y(z,z' \in [Z] \wedge x \in [X] \wedge y \in [Y] \wedge dist(x,y) = dist(z,z'))$.

Proof. See Appendix B.3. □

In D_α (i.e., the natural domain of **T1**), the interpretation of CCon_1 is not equivalent to the interpretation of CCon given in Table 1. For example, let $n = 1$ and take $X = (2,3)$, $Y = (5,6)$, and $Z = (0,1) \cup (7,8)$. Then, $\mathrm{dist}(X,Y) \leq \mathrm{diam}(Z)$, but $\neg\exists z,z',x,y(z,z' \in Z \wedge x \in X \wedge y \in Y \wedge \mathrm{dist}(x,y) = \mathrm{dist}(z,z'))$. In addition, while the interpretation in Table 1 satisfies all the axioms given by De Laguna on the *can connect* primitive in all the structures, the interpretation of CCon_1 does not satisfy (in Φ_α) the following De Laguna axiom:
$\exists x,y(\mathsf{CCon}(a,x,y) \wedge \neg\mathsf{CCon}(b,x,y)) \rightarrow \neg\exists z,\upsilon(\mathsf{CCon}(b,z,\upsilon) \wedge \neg\mathsf{CCon}(a,z,\upsilon))$.

A counterexample in D_α (for $n = 1$) is obtained by taking $A = (3,5)$, $B = (0,1) \cup (7,8)$, $X = (2,3)$, $Y = (5,6)$, $Z = (-1,0)$, and $V = (8,9)$.

We conclude that (DCC1) does not capture the De Laguna's *can connect* primitive, and therefore, in our conceptual comparison, we will not make use of this definition.

5.1.2. Definitions in T2. *Primitives of* **T2**: P, SR, CG. The definition (DC2), given below, uses the relationship CNC. This has been discussed in Definitions in **T1** section and is adopted in **T2** without changes with respect to Tarski's work. Of course, the correctness of CNC in this theory depends on the correctness of definition (DS2), which establishes what counts as a sphere in this theory.

Explicit definitions furnished in **T2**:
(DS2) $\mathsf{S}^*_2(x) =_{\mathrm{def}} \mathsf{SR}(x) \wedge \forall y,z((\mathsf{CG}(x,y) \wedge \mathsf{PO}(x,y) \wedge \mathsf{DIF}(z,x,y)) \rightarrow \mathsf{SR}(z))$;
(DC2) $\mathsf{C}_2(x,y) =_{\mathrm{def}} \exists z(\mathsf{S}^*_2(z) \wedge \forall z'(\mathsf{CNC}(z',z) \rightarrow (\mathsf{O}(z',x) \wedge \mathsf{O}(z',y))))$.

Proposition 4. $[\![\mathsf{S}^*_2(x)]\!]_\beta = [\![\mathsf{S}(x)]\!]$ (Φ_β *is the natural environment structure of* **T2**) *and* $[\![\mathsf{S}^*_2(x)]\!]_{\alpha,\gamma,\delta} \neq [\![\mathsf{S}(x)]\!]$.

Proof. See Appendix B.4. □

Proposition 5. $[\![\mathsf{C}_2(x,y)]\!] = [\![\mathsf{C}(x,y)]\!]$ *provided* $[\![\mathsf{S}^*_2(x)]\!] = \mathsf{S}(x)$.

Proof. It follows from the proof of Proposition 2. □

Bennett *et al.* (2000a) propose a definition of S based on P and CG together with an attempt to provide semantic equivalence in the domain of open regular regions. We are not going to consider this definition in our comparison because, unfortunately, it fails to capture the notion of sphere. The interested reader can easily verify that the definition in that paper does not rule out nonspherical regions like Reuleaux polytopes.[17]

5.1.3. Definitions in T4. *Primitive of* **T4**: CCon. *Explicit definitions furnished in* **T4**:
(DC4*) $\mathsf{C}^*_4(x,y) =_{\mathrm{def}} \forall z(\mathsf{CCon}\,(z,x,y))$;

[17] A Reuleaux polytope in R^2 is the region obtained by intersecting the 3 discs centered at the vertices of an equilateral triangle with radius (of length) equal to (the length of) the side of the triangle itself.

(DP4*) $\mathsf{P}^*_4(x, y) =_{\text{def}} \forall z(\mathsf{C}^*_4(z, x) \to \mathsf{C}^*_4(z, y))$;
(DP4$^+$) $\mathsf{P}^+_4(x, y) =_{\text{def}} \forall z, w(\mathsf{CCon}(w, z, x) \to \mathsf{CCon}(w, z, y))$;
(DCl4) $\mathsf{Closer}_4(z, x, y) =_{\text{def}} \exists a(\mathsf{CCon}(a, z, x) \wedge \neg\mathsf{CCon}(a, z, y))$.

Proposition 6. $[\![\mathsf{C}^*_4(x, y)]\!]_{\beta,\delta} = [\![\mathsf{C}(x, y)]\!]$ *(Φ_δ is the open counterpart of Φ_θ, i.e., of the natural environment structure of* **T4**) *and* $[\![\mathsf{C}^*_4(x, y)]\!]_{\alpha,\gamma} \neq [\![\mathsf{C}(x, y)]\!]$.

Proof. See Appendix B.5. □

Proposition 7. $[\![\mathsf{C}^*_4(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{P}(x, y)]\!], [\![\mathsf{P}^+_4(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{P}(x, y)]\!]$, *and* $[\![\mathsf{Closer}_4(z, x, y)]\!]_{\alpha-\delta} = [\![\mathsf{Closer}(z, x, y)]\!]$.

Proof. See Appendix B.6. □

5.1.4. Definitions in T6. *Primitives of* **T6**: C, ConvH.
Explicit definitions furnished in **T6**:[18]
(DP6) $\mathsf{P}_4(x, y) =_{\text{def}} \forall z(\mathsf{C}(z, x) \to \mathsf{C}(z, y))$;
(DCM6) $\mathsf{Compl}_6(y, x) =_{\text{def}} \forall z(\mathsf{C}(z, y) \leftrightarrow \neg\mathsf{IP}(z, x))$;[19]
(DSR6) $\mathsf{SR}_6(x) =_{\text{def}} \forall y, z, w((\mathsf{SUM}(x, y, z) \wedge \mathsf{Compl}_6(w, x)) \to \exists v(\mathsf{SC}(v) \wedge \mathsf{O}(v, y)$
$\wedge\, \mathsf{O}(v, z) \wedge \neg\mathsf{C}(v, w)))$.[20]

Proposition 8. $[\![\mathsf{P}_6(x, y)]\!]P_{\alpha-\delta} = [\![\mathsf{P}(x, y)]\!]$ *and* $[\![\mathsf{SR}_6(x)]\!]\mathsf{SR}_{\alpha-\delta} = [\![\mathsf{SR}(x)]\!]$.

Proof. See Appendix B.7. □

5.2. Dispensable primitives. Using extensively the definitions and results of the previous sections, we now investigate if some primitives of a theory **Ti** can be defined (in all $\Phi_{\alpha-\delta}$) on the basis of the other primitives of the same theory **Ti**. This ‘internal reduction’ points out redundancies and reduces the steps needed to compare the theories.

Proposition 9. *In* **T2**, *we can use* (DSR6) *to define the relation* SR.

Proof. Directly from Proposition 8 and the fact that all the predicates used in (DSR6) are definable in **T2** with the same interpretation. □

In **T3**, on the basis of the primitive Conj, we can define the parthood relation:
(DC3*) $\mathsf{C}^*_3(x, y) =_{\text{def}} \forall z(\mathsf{Conj}(z, z, x, y))$;
(DP3) $\mathsf{P}_3(x, y) =_{\text{def}} \forall z(\mathsf{C}^*_3(z, x) \to \mathsf{C}^*_3(z, y))$.

Proposition 10. $[\![\mathsf{P}_3(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{P}(x, y)]\!]$.

Proof. See Appendix B.8. □

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[18] The predicates O, IP, SUM, and SC in the definitions (DCM6) and (DSR6) are defined using P_6. Because of Propositions 1 and 8, their definitions coincide with the original ones. For this reason, we do not introduce new symbols for these predicates.

[19] In the original papers, the complement is introduced as a primitive function. Here we adopt the standard formulation of the complement as a relation.

[20] This predicate is called ‘manifold’ in Cohn (1995).

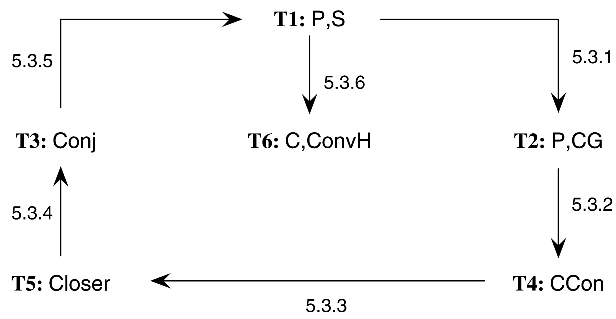

Fig. 1. Definitional links between mereogeometries (we report the primitives of each thoery). Here '**Ti** $\rightarrow$ **Tj**' means 'theory **Tj** is a $\Phi_{\alpha-\theta}$-subtheory of **Ti**,' i.e., in $\Phi_{\alpha-\theta}$, all the primitives **Tj** can be defined on the basis of the primitives of **Ti** (the labels indicate the section in which the proof is given).

In **T5**, on the basis of the primitive Closer, we can define the connection relation:
(DC5*) $\mathsf{C}^*_5(x, y) =_{\text{def}} \neg\exists\mathsf{Closer}(x, z, y))$;
(DP5) $\mathsf{P}_5(x, y) =_{\text{def}} \forall z(\mathsf{C}^*_5(z, x) \rightarrow \mathsf{C}^*_5(z, y))$;
(DFD5) $\mathsf{FD}_5(x) =_{\text{def}} \exists z(\forall x', x''((\mathsf{P}_5(x', x) \wedge \mathsf{P}_5(x'', x)) \rightarrow \mathsf{Closer}(x', x'', z)))$;
(x has finite diameter)
(DC5) $\mathsf{C}_5(x, y) =_{\text{def}} \exists z, w(\mathsf{FD}_5(z) \wedge \mathsf{FD}_5(w) \wedge \mathsf{P}_5(z, x) \wedge \mathsf{P}_5(w, y) \wedge \mathsf{C}^*_5(z, w))$.

Proposition 11. $[\![\mathsf{C}_5(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{C}(x, y)]\!]$.

Proof. See Appendix B.9. □

5.3. Linking via explicit definitions. In this section, we show how the mereogeometries **T1**–**T6** are related in the structures $\Phi_{\alpha-\theta}$. The connections are illustrated in Figure 1, which shows our main result. At the end of this section, we will be able to conclude that these mereogeometries, with the exception of **T6**, are actually $\Phi_{\alpha-\theta}$-equivalent.

5.3.1. From T1 to T2. By Proposition 2, in **T1**, (DC1) defines C in all the structures; therefore, we can use all the relations defined in Proposition 1. We use the additional relations $\mathsf{ID}(z, x, y)$ (x and y are internally diametrical with respect to z) and $\mathsf{CNC}(x, y)$ (x is concentric with y), which were introduced by Tarski (1956a). As done before, we report only the interpretations. A full description and the related proof of correctness can be found in Tarski (1956a).

$[\![\mathsf{ID}(z,x, y)]\!]_{\alpha-\delta} = \mathrm{Int}D(Z, X, Y)$

(the centers of z, x, y are aligned and z is the minimum sphere containing x, y)

$[\![\mathsf{CNC}(x, y)]\!]_{\alpha-\delta} = \exists c,r_1,r_2\ (X_1 = \mathrm{ball}(c, r_1) \wedge X_2 = \mathrm{ball}(c, r_2))$.
Using all these relations, we can define when 2 regions are congruent:
$\mathsf{CG}_1(x, y) =_{\text{def}} \forall z(\Sigma\ \mathsf{SS}(z) \rightarrow \exists z'(\Sigma\mathsf{CG}(z, z') \wedge \forall s,s'((\mathsf{MSP}(s, z) \wedge \mathsf{MSP}(s', z') \wedge \mathsf{SCG}(s, s')) \rightarrow$
$((\mathsf{P}(s, x) \leftrightarrow \mathsf{P}(s', y)) \wedge \mathsf{P}(s, y) \leftrightarrow \mathsf{P}(s', x)) \wedge \mathsf{PO}(s, x) \leftrightarrow \mathsf{PO}(s', y)) \wedge \mathsf{PO}(s, y) \leftrightarrow \mathsf{PO}(s', x))))))$

(x and y are congruent regions),

where

$\mathsf{SCG}(x, y) =_{\text{def}} \mathsf{S}(x) \wedge \mathsf{S}(y) \wedge (x = y \ \vee \ \exists z,w(\mathsf{CNC}(z, w) \wedge \mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y) \ \wedge \mathsf{TPP}(x, w) \wedge \mathsf{TPP}(y, w)))$

(x and y are congruent spheres)

$\mathsf{EqD}(x, y, x', y') =_{\text{def}} \mathsf{SCG}(x, x') \wedge \mathsf{SCG}(y, y') \wedge \neg\mathsf{P}(x, y) \wedge \neg\, \mathsf{P}(y, x) \wedge \neg\, \mathsf{P}(x', y') \wedge \neg\, \mathsf{P}(y', x') \wedge \exists z,w(\mathsf{ID}(z, x, y) \wedge \mathsf{ID}(w, x', y') \wedge \mathsf{SCG}(z, w))$

(x, x' are congruent spheres and so are y, y'; x and y are not one part of the other, analogously x' and y'; the centers of x, y and those of x', y' are equidistant)

$\mathsf{MSP}(x, y) =_{\text{def}} \mathsf{S}(x) \wedge \mathsf{P}(x, y) \wedge \forall z((\mathsf{S}(z) \wedge \mathsf{PP}(x, z)) \rightarrow \neg\mathsf{P}(z, y))$ (x is a maximal sphere contained in y)

$\Sigma\mathsf{SS}(x) =_{\text{def}} \forall y(\mathsf{P}(y, x) \rightarrow \exists s(\mathsf{MSP}(s, x) \wedge \mathsf{O}(s, y))) \wedge$
$\forall u,w((\mathsf{MSP}(u, x) \wedge \mathsf{MSP}(w, x) \wedge u \neq w) \rightarrow \neg\mathsf{SCG}(u, w))$

(x is the sum of a set of pairwise noncongruent spheres)

$\Sigma\mathsf{CG}(x, y) =_{\text{def}} \Sigma\mathsf{SS}(x) \wedge \Sigma\mathsf{SS}(y) \wedge$
$\forall s(\mathsf{MSP}(s, x) \rightarrow \exists s'(\mathsf{MSP}(s', y) \wedge \mathsf{SCG}(s, s'))) \wedge$
$\forall s(\mathsf{MSP}(s, y) \rightarrow \exists s'(\mathsf{MSP}(s', x) \wedge \mathsf{SCG}(s, s'))) \wedge$
$\forall s,u,s',u\ (\mathsf{MSP}(s, x) \wedge \mathsf{MSP}(u, x) \wedge \mathsf{MSP}(s', y) \wedge \mathsf{MSP}(u', y) \wedge \mathsf{SCG}(s, s')$
$\wedge\ \mathsf{SCG}(u, u'))$
$\rightarrow \mathsf{EqD}(s, u, s', u')$

(regions x and y are congruent and they are the sum of 2 equivalent sets of pairwise noncongruent spheres).

Lemma 1. **T2** *is a* $\Phi_{\alpha-\delta}$*-subtheory of* **T1**.

Proof. We need to prove that $[\![\mathsf{CG}_1(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{CG}(x, y)]\!]$. See Appendix C.1. □

5.3.2. From T2 to T4. In Appendix C.2, we show that $[\![\mathsf{C}^*_2(x,y)]\!]_{\alpha-\delta} = (\mathrm{dist}(X, Y) = 0)$. From this, we can define parthood by $\mathsf{P}_2(x,y) =_{\text{def}} \forall z(\mathsf{C}^*_2(z, x) \rightarrow \mathsf{C}^*_2(z, y))$. The proof that this definition is correct is as the proof of Proposition 10. Then, Proposition 1 gives us all the mereological relations used in the following definitions.

$\mathsf{CCon}_2(c, x, y) =_{\text{def}} \forall a,b(\mathsf{LDist}_2(a, b, x, y) \rightarrow \exists z(\mathsf{SCDiam}_2(z, c) \wedge \mathsf{O}(z, a) \wedge \mathsf{O}(z, b)))$,

where:

$\mathsf{C}^*_2(x, y) =_{\text{def}} \forall z\exists z'(\mathsf{CG}(z', z) \wedge \mathsf{O}(z', x) \wedge \mathsf{O}(z', y))$;

$\mathsf{SC}^*_2(x) =_{\text{def}} \forall y,z(\mathsf{SUM}(x, y, z) \rightarrow \mathsf{C}^*_2(y, z))$; ($x$ is w-weakly connected)

$\mathsf{LEDiam}_2(x, y) =_{\text{def}} \mathsf{SC}^*_2(y) \wedge \forall a,b((\mathsf{P}(a, x) \wedge \mathsf{P}(b, x)) \rightarrow \exists y'(\mathsf{CG}(y', y) \wedge \mathsf{O}(y', a) \wedge \mathsf{O}(y', b)))$;

(the diameter of x is less than or equal to the diameter of y and y is w-weakly connected)

$\mathsf{SCDiam}_2(x, y) =_{\text{def}} \mathsf{SC}^*_2(x) \wedge \forall z((\mathsf{P}(y, z) \wedge \mathsf{SC}^*_2(z)) \rightarrow \mathsf{LEDiam}_2(x, z))$;

(the diameter of x is less than or equal to the diameter of y and x is w-weakly connected)

$\mathsf{LDist}_2(x, y, x', y') =_{\text{def}} \exists a(\mathsf{SC}^*_2(a) \wedge \mathsf{O}(a, x) \wedge \mathsf{O}(a, y) \wedge \forall a'(\mathsf{CG}(a', a) \rightarrow (\neg\mathsf{O}(a', x') \vee \neg\mathsf{O}(a', y'))))$

(the distance of x from y is strictly smaller than the distance of x' from y').

Lemma 2. **T4** *is a* $\Phi_{\alpha-\delta}$*-subtheory of* **T2**.

Proof. We need to prove that $[\![\mathsf{CCon}_2(c, x, y)]\!]_{\alpha-\delta} = [\![\mathsf{CCon}(c, x, y)]\!]$. See Appendix C.2. □

5.3.3. From T4 to T5. In **T5**, the relation C is dispensable (see Proposition 11); therefore, it is sufficient to provide an explicit definition of Closer in terms of CCon. For this, we use the definition (DCl4) of Definitions in **T4** section:

(DCl4) $\mathsf{Closer}_4(z, x, y) =_{\mathrm{def}} \exists a(\mathsf{CCon}(a, z, x) \wedge \neg\mathsf{CCon}(a,z,y))$.

Lemma 3. **T5** *is a* $\Phi_{\alpha-\delta}$*-subtheory of* **T4**.

Proof. We need to prove that $[\![\mathsf{Closer}_4(z, x, y)]\!]_{\alpha-\delta} = [\![\mathsf{Closer}(z, x, y)]\!]$. This follows from Proposition 7. □

5.3.4. From T5 to T3. Here the explicit definitions we need are more complex. The main reason is that we cannot find a way to split the definitions in pieces, which correspond to intuitive or already known notions. So, we end up with a relatively long set of conditions that, taken together, provide the correct constraints, although from such a set of conditions one has little hope to recover the intuition about the defined notion.

In the specific case we deal with in this section, we further split the definition of equidistance (EqD^*) in 2 cases depending on the dimension of the domain. This is needed for domains of finite regions, like Φ_γ and Φ_δ. Thus, we provide 2 definitions of EqD^*: one for the 1-dimensional domains and one for the others.

Since in **T3** the relation P is dispensable (Proposition 10) and in **T5** both C and P are definable from Closer (Proposition 11 and the definition (DP) of Proposition 1), the following turns out to be an explicit definition of Conj in terms of Closer:

$$\begin{aligned}&\mathsf{Conj}_5(x, y, x', y') =_{\mathrm{def}} \exists a,b,a',b'(\mathsf{SR}(a) \wedge \mathsf{SR}(b) \wedge \mathsf{SR}(a') \wedge \mathsf{SR}(b') \wedge\\ &\mathsf{C}(a, x) \wedge \mathsf{C}(b, y) \wedge \mathsf{C}(a', x') \wedge \mathsf{C}(b', y') \wedge \mathsf{EqD}^*(a, b, a', b') \wedge\\ &\forall p_{\mathrm{a}}, p_{\mathrm{b}}((\mathsf{P}(p_{\mathrm{a}}, a) \wedge \mathsf{P}(p_b, b) \wedge \mathsf{EqD}^*(p_a, p_b, a, b)) \rightarrow (\mathsf{C}(p_a, x) \wedge \mathsf{C}(p_b, y)) \wedge\\ &\forall p'_a, p'_b((\mathsf{P}(p'_a, a') \wedge \mathsf{P}(p'_b, b') \wedge \mathsf{EqD}^*(p'_a, p'_b, a',b')) \rightarrow (\mathsf{C}(p'_a, x') \wedge \mathsf{C}(p'_b, y')),\end{aligned}$$

where:

SR, given by (DSR6) in Definitions in **T6** section, is defined using C and P only;

FD_5 is defined in terms of Closer by (DFD5) in Dispensable Primitives section;

$\mathsf{Eq}(z, x, y) =_{\mathrm{def}} \neg\mathsf{Closer}(z, x, y) \wedge \neg\mathsf{Closer}(z,y,x)$; ($z$ is equidistant from x and y)

and EqD^* has different definitions in different domains. Then, in $\Phi_{\alpha-\beta}$ for R^1 and in $\Phi_{\alpha-\delta}$ for $\mathrm{R}^{n>1}$, we take

(a) $\mathsf{EqD}^*(x, y, x', y') =_{\mathrm{def}} \mathsf{FD}_5(x) \wedge \mathsf{FD}_5(y) \wedge \mathsf{FD}_5(x') \wedge \mathsf{FD}_5(y') \wedge$
$\exists z,z'(\mathsf{Eq}(x, y, z) \wedge \mathsf{Eq}(x', y', z') \wedge \mathsf{Eq}(z, x, z') \wedge \mathsf{Eq}(z', x', z))$
(x is as close to y as x' is to y').

In $\Phi_{\gamma-\delta}$ for R^1, we take

(b) $\mathsf{EqD}^*(x, y, x', y') =_{\mathrm{def}} (\mathsf{C}(x, y) \wedge \mathsf{C}(x', y')) \vee \exists z,z'(\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y) \wedge \mathsf{EC}(z', x') \wedge$
$\mathsf{EC}(z', y') \wedge \mathsf{CG}^*(z, z'))$,

where

$\mathsf{CG}^{\mathsf{s}}(x, y) =_{\text{def}} \mathsf{FD}_5(x) \wedge \mathsf{FD}_5(y) \wedge \neg\mathsf{C}(x, y) \wedge \exists z_1,z_2,z_3(\mathsf{EC}(z_2, x) \wedge \mathsf{EC}(z_2, y) \wedge \mathsf{EC}(z_1, x) \wedge \neg\, \mathsf{C}(z_1, z_2) \wedge \mathsf{EC}(z_3, y) \wedge \neg\, \mathsf{C}(z_3, z_2) \wedge \mathsf{Eq}(z_2, z_1, z_3))$;

$\mathsf{CG}^*(x, y) =_{\text{def}} \mathsf{FD}_5(x) \wedge \mathsf{FD}_5(y) \wedge$
$(x = y \vee$
$(\mathsf{PO}(x, y) \wedge \exists z_1,z_2(\mathsf{DIF}(z_1, x, y) \wedge \mathsf{DIF}(z_2, y, x) \wedge \mathsf{CG}^{\mathsf{s}}(z_1, z_2))) \vee$
$(\mathsf{EC}(x, y) \wedge \neg\exists z((\mathsf{PP}(z, x) \wedge \mathsf{CG}^{\mathsf{s}}(z, y)) \vee (\mathsf{PP}(z, y) \wedge \mathsf{CG}^{\mathsf{s}}(z, x)))) \vee$
$(\neg\mathsf{C}(x, y) \wedge \mathsf{CG}^{\mathsf{s}}(x, y)))$.[21]

Lemma 4. **T3** *is a* $\Phi_{\alpha-\delta}$*-subtheory of* **T5**.

Proof. We need to prove that $[\![\mathsf{Conj}_5(x, y, x', y')]\!]_{\alpha-\delta} = [\![\mathsf{Conj}(x, y, x', y')]\!]$. See Appendix C.3. □

5.3.5. From T3 to T1. Recall from Dispensable Primitives section that $\mathsf{C}_3^*(x,y)$ is defined using Conj via (DC3*), i.e.,

(DC3*) $\mathsf{C}_3^*(x,y) =_{\text{def}} \forall z(\mathsf{Conj}(z, z, x, y))$.

P (and therefore also SUM) is definable using C_3^* (see Proposition 10) and $[\![\mathsf{C}_3(x,y)]\!]_{\alpha-\delta} = [\![\mathsf{C}(x,y)]\!]$ (see Appendix C.4). Therefore, we can rely on several definitions introduced in Proposition 1 (e.g., SC and TPP) and on the definition of SR introduced in Proposition 8.

Using these relations, we can give an explicit definition of sphere in terms of Conj:

$$\mathsf{S}_3(x) =_{\text{def}} \mathsf{FD}_3(x) \wedge \mathsf{SR}(x) \wedge \forall a(\mathsf{LEDiam}_3(a, x) \rightarrow \exists b(\mathsf{LEDiam}_3(b, a) \wedge \mathsf{P}(b, x) \wedge \forall c,d(\mathsf{EC}(c, x) \wedge \mathsf{EC}(d, x)) \rightarrow (\mathsf{Conj}(b, c, b, d))),$$

where

$\mathsf{SC}_3^*(x) =_{\text{def}} \forall y, z(\mathsf{SUM}(x, y, z) \rightarrow \mathsf{C}_3^*(y, z))$;
(x is w-weakly connected)

$\mathsf{FD}_3(x) =_{\text{def}} \exists x', y, z(\mathsf{SC}_3^*(x') \wedge \mathsf{P}(x, x') \wedge \neg\mathsf{Conj}(x', x', y, z))$;
(x has finite diameter)

$\mathsf{C}_3(x, y) =_{\text{def}} \exists z, w(\mathsf{FD}_3(z) \wedge \mathsf{FD}_3(w) \wedge \mathsf{P}(z, x) \wedge \mathsf{P}(w, y) \wedge \mathsf{C}_3^*(z, w))$

$\mathsf{LEDiam}_3(x,y) =_{\text{def}} \mathsf{SR}(x) \wedge \mathsf{SR}(y) \wedge \forall a,b((\mathsf{P}(a,x) \wedge \mathsf{P}(b,x)) \rightarrow \exists a',b'(\mathsf{P}(a',y) \wedge \mathsf{P}(b',y) \wedge \mathsf{Conj}(a,b,a',b')))$
(the diameter of x is less than or equal to the diameter of y, and x and y are connected)

Lemma 5. **T1** *is a* $\Phi_{\alpha-\delta}$*-subtheory of* **T3**.

Proof. We need to prove that $[\![\mathsf{S}_3(x)]\!]_{\alpha-\delta} = [\![\mathsf{S}(x)]\!]$. See Appendix C.4. □

5.3.6. From T1 to T6. By Proposition 2, in **T1**, (DC1) defines C for all the structures. Therefore, it is sufficient to provide an explicit definition of ConvH in terms of P and S. We use the additional relation BTW, which was introduced by Tarski (1956a). As done before, we report only the interpretation. A full description and the related proof of correctness can be found in Tarski (1956a).

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[21] In this definition, some conditions are redundant like $\neg\mathsf{C}(x, y)$. We include them in the attempt to improve the readability of the formula.

$[\![\mathsf{BTW}(x_1, x_2, x_3)]\!]_{\alpha-\delta} = \exists c_1,c_2,c_3,r_1,r_2,r_3(\mathrm{Btw}(c_1, c_2, c_3) \wedge X_1 = \mathrm{ball}(c_1, r_1) \wedge X_2 = \mathrm{ball}(c_2, r_2) \wedge X_3 = \mathrm{ball}(c_3, r_3))$.

Using the above BTW, we can define

$\mathsf{ConvH}_1(x, y) =_{\mathrm{def}} \mathsf{Conv}(x) \wedge \mathsf{P}(y, x) \wedge \neg\exists z(\mathsf{Conv}(z) \wedge \mathsf{P}(y, z) \wedge \mathsf{PP}(z, x))$,

where
$\mathsf{Conv}(x) =_{\mathrm{def}} \forall s_1,s_2,s_3((\mathsf{P}(s_1, x) \wedge \mathsf{P}(s_2, x) \wedge \mathsf{BTW}(s_3, s_1, s_2)) \rightarrow \mathsf{O}(s_3, x))$.

Lemma 6. **T6** *is a* $\Phi_{\alpha-\delta}$*-subtheory of* **T1**.

Proof. $[\![\mathsf{ConvH}_1(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{ConvH}(x, y)]\!]$. See Appendix C.5. □

5.3.7. The main theorem. Now, we can state the main result of this paper:

Main theorem

(a) **T1**–**T5** *are* $\Phi_{\alpha-\theta}$*-equivalent*;
(b) **T6** *is a* $\Phi_{\alpha-\theta}$*-subtheory of* **T1–T5**;
(c) **T1**–**T4** *are conceptually equivalent*.

Proof.

(a) For $\Phi_{\alpha-\delta}$, the thesis follows from Lemmas 1–5. For the structures $\Phi_{\varepsilon-\theta}$, it follows from our argument in Environment Structures section together with the results of Lemmas 1–5.
(b) From Lemma 6 and (a).
(c) From (a) and Definition 4. □

Note that we do not put constraints on the dimension of the space. Indeed, the result is valid in R^n for any positive n. On the other hand, the result relies on the properties of the considered domains, and it might be hard, if possible at all, to extend it to other domains. For example, it is known that P cannot be defined from C using the definition (DP) in Verifying the Given Explicit Definitions section when dealing with atomic theories (Masolo & Vieu, 1999; Randell & Cohn, 1992). A similar result holds between C and SR as given in **T2**.

6. Final comments. As we have pointed out in the introduction, a major motivation for this comparison of mereogeometries is the need of evaluating the strength of the mereogeometrical systems in the literature. It is known from the work of Tarski that system **T1** can be used to capture the full system of Euclidean geometry by defining, in second-order logic, points to be collections of concentric spheres. This result suggests that theory **T1** is perhaps the strongest system we can look for while remaining within the realm of (region-based) geometry. The most relevant systems in the literature that we have analyzed are formally equivalent to **T1** in the sense of Conceptual Comparison section. We take this fact as evidence that all these theories capture the 'same' notion of (mereo)geometry and that the strength of other systems should be measured with respect to these.

Definition 6. *A* full mereogeometry *is a theory that is conceptually equivalent to* ***T1***.

Here is an immediate consequence of the main theorem

Corollary 1. **T1**–**T4** *are full mereogeometries.*

Our comparison does not establish the exact relationship between a full mereogeometry and theory **T6**. It has been argued in Cohn (1995) that the predicates C and ConvH do not suffice to obtain what we call here a full mereogeometry. Furthermore, the primitive ConvH, at least when interpreted in R^n, seems to be naturally related to a (restricted) application of the Btw relation (see Basic Notions in R^n section), that is, to a relation that alone is too weak to capture Euclidean geometry (Tarski, 1956b). These observations make us to believe that **T6** cannot be as strong as **T1**. This is consistent with the results in Davis (2006) and Davis *et al.* (1999) and matches the conjecture 'Mereology + Convexity = Affine Geometry' in Pratt-Hartmann (1999). However, we have no direct proof of this and leave the issue as an open question.

Conjecture. **T6** is not a full mereogeometry.

We think that the conceptual analysis of mereogeometries presented in this work, even with the limits discussed in Environment Structures section, puts order on the relationship among important theories in the literature. In particular, the main theorem states that *in the given environment structures*, the theories **T1–T5** have the same expressive power. This means that, leaving aside computational issues, there is no real difference among these theories and that, for applicative concerns, the choice of which system to adopt can be safely based on nonlogical issues like, for instance, cognitive and modeling adequacy.

We remark here once more why our analysis is not conclusive about the classification of theory **T5** as a full mereogeometry. We have seen that this theory is formally equivalent to **T1–T4** in the frameworks we considered. Nonetheless, our analysis also considers nonformal aspects among which there is the natural domain of interpretation for the theory. Theory **T5** is introduced with a natural domain that we have not considered, and we have no proof that this theory is equivalent to the others in an environment with such a domain. Thus, as of now, we cannot claim that **T5** itself is a full mereogeometry. What we can say is that if someone wants to use the formal system **T5** within one of the domains we considered, call this theory **T5′**, then from our result it follows that theory **T5′** is a full mereogeometry to all effects.

It is important to note that the definitions we have studied in this paper are all stated in a first-order language. Therefore, they can be applied to furnish explicit definitions between (fragments of) the theories. As we have seen, in some cases, these definitions are quite complex. The complexity may increase even further if we look for a direct connection between those theories that we did not link explicitly. For example, the definition of the primitives of **T1** in terms of the primitives of **T5** is given indirectly: in the first step, we define the primitives of **T3** in **T5**, and then, we use these to define those of **T1**. A complete analysis of these definitions focusing on the complexity of the formulas could highlight important aspects from both the conceptual and the applicative points of view.

Finally, since **T1** is semantically complete with respect to its natural model with domain D_α, our explicit definitions in the subsections From **T1** to **T2**, From **T2** to **T4**, From **T4** to **T5**, From **T5** to **T3**, and From **T3** to **T1** provide a simple way to obtain a semantically complete axiomatization of all the theories **T2**–**T5** in the domain D_α. This result is particularly relevant since, as we have seen, many systems are presented in the literature with a partial axiomatization only.

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8. Appendix A.

8.1. Basic topological and geometrical lemmas.

Lemma L.1. *Given a regular set X in* R^n, *then Conx(X)* $\rightarrow$ *WConx*(X) $\rightarrow$ *WWConx*(X), *while none of the converse implications hold in general. Furthermore, (WWConx*$(X)\wedge$ *diam*$(X) < +\infty) \rightarrow$ *WConx*(X).

Proof. For X empty, there is nothing to prove. Otherwise, the first claim holds from the definitions since $A^\circ \cap B^\circ \neq \emptyset$ implies $[A] \cap [B] \neq \emptyset$ and the latter implies dist$(A, B) = 0$.

WConx$(X) \rightarrow$ Conx(X) fails. For a counterexample in R^2, take $X = A \cup B$, where $A = \{(x, -y)|y > 0, x > 0\}$ and $B = \{(x, y)|y > 0, x > 0\}$. WWConx$(X) \rightarrow$ WConx(X) fails as well. For a counterexample in R^2, take $X = A \cup B$, where $A = \{(x, y)|y > 1/x > 0\}$ and $B = \{(x, y)\ |x > 0, y < 0\}$.

Finally, let WWConx(X) with diam$(X) < +\infty$. Fix any pair A, B satisfying the definition for WWConx(X) and consider a sequence (a_i) in A and a sequence (b_i) in B such that $\lim_{i\rightarrow\infty}$ dist$(a_i, b_i) = 0$ (such sequences exist since dist$(A, B) = 0$). Since diam$(X) < +\infty$, there exist a,$b \in \mathrm{R}^n$ and subsequences (a_j) and (b_j) of (a_i) and (b_i), respectively, such that $a = \lim_{j\rightarrow\infty} a_j$ and $b = \lim_{j\rightarrow\infty} b_j$. Clearly, $a \in [A]$ and $b \in [B]$, and from dist$(a, b) = \lim_{j\rightarrow\infty}$ dist$(a_j, b_j) = 0$, we conclude $a = b$, that is, $[A] \cap [B] \neq \emptyset$ □

Lemmas L.2–L.6. *Let X and Y be arbitrary sets in a topological space, then*:
L.2 $X^\circ \subseteq X \subseteq [X]$.
L.3 $[X \cup Y] = [X] \cup [Y]$ *and* $[X \cap Y] \subseteq [X] \cap [Y]$.
L.4 $(X \cap Y)^\circ = X^\circ \cap Y^\circ$ *and* $X^\circ \cup Y^\circ \subseteq (X \cup Y)^\circ$.
L.5 If $X \subseteq Y$, *then* $X^\circ \subseteq Y^\circ$ *and* $[X] \subseteq [Y]$.
L.6 If $[X] \cap [Y] = \emptyset$, *then* $(X \cup Y)^\circ = X^\circ \cup Y^\circ$.

Proof. Lemmas L.2–L.5 are basic topological results (see for instance Munkres, 2000).

Regarding L.6: from L.4, we know $X^\circ \cup Y^\circ \subseteq (X \cup Y)^\circ$; we need to show $X^\circ \cup Y^\circ \supseteq (X \cup Y)^\circ$. If X or Y is empty, there is nothing to prove. Assume that they are both nonempty and $x \in (X \cup Y)^\circ$, then $x \in X \cup Y$. Suppose $x \in X$. If $x \notin X^\circ$, then $x \in X^\circ \cup Y^\circ$, and we are done. If $x \notin X^\circ$, i.e., $x \in \partial(X)$, then $x \notin [Y]$ since $[X] \cap [Y] = \emptyset$. Thus, there exists a neighborhood of x, say $I(x)$, such that $I(x) \cap Y = \emptyset$. From $x \in \partial(X)$ and $x \notin [Y]$, $I(x) \cap (X \cup Y) \neq \emptyset$ and $I(x) \cap \sim (X \cup Y) \neq \emptyset$. This happens for any neighborhood of x contained in $I(x)$, thus $x \in \partial(X \cup Y)$, contradicting the hypothesis $x \in (X \cup Y)^\circ$. Finally, we have $x \notin \partial(X)$. Thus, $x \in X$ implies $x \in X^\circ$. One can prove analogously that $x \in Y$ implies $x \in Y^\circ$. From these results, $x \in (X \cup Y)^\circ$ implies $x \in X^\circ \cup Y^\circ$, and we are done. □

Lemma L.7. *Let X and Y be open regular sets in a topological space T, then* $X \cap Y$, $[X \cup Y]^\circ$, *and* $\sim [X]$ *are open regular sets. Let X and Y be closed regular sets in a topological space T, then* $[(X \cap Y)^\circ]$, $X \cup Y$, *and* $\sim (X^\circ)$ *are closed regular sets.*

Proof. Directly from the fact that regular open sets form a Boolean algebra with $1 = \mathrm{T}$, $0 = \emptyset$, $X \cdot Y = X \cap Y$, $X + Y = [X \cup Y]^\circ$, and $-X = \sim [X]$ (Biacino & Gerla, 1991). Analogously for the regular closed sets. □

Lemma L.8. *Given 2 nonempty regular sets X and Y in* R^n*:*
dist$(X^\circ, Y^\circ) =$ *dist*$(X, Y) =$ *dist*$([X], [Y])$.

Proof. From L.2 and from the definition of distance, we have dist($[X]$, $[Y]$) $\leq$ dist(X, Y) $\leq$ dist(X°, Y°). For the other direction, suppose (by contradiction) that dist($[X]$, $[Y]$) $<$ dist(X°, Y°), i.e., i = inf{dist(x, y)| $x \in [X] \wedge y \in [Y]$} $<$ inf{dist(x, y)|$x \in X^\circ \wedge y \in Y^\circ$} = i'. Then, there exists $d > 0$ such that $i' = i + d$. From the definition of distance between sets, for all $\varepsilon > 0$ there exist $x \in [X]$ and $y \in [Y]$ such that dist(x, y) $< i + \varepsilon$, fix $\varepsilon = d/3$, then dist(x, y) $< i'$. Thus, ball(x, ε) $\cap X^\circ = \emptyset$ or ball(y,ε) $\cap Y^\circ = \emptyset$ (otherwise, we could find points $x' \in X^\circ$ and $y' \in Y^\circ$ with dist(x', y') $<$ i′). From this result, $x \notin [X]$ or $y \notin [Y]$, a contradiction. Then, dist($[X]$, $[Y]$) $\geq$ dist(X°, Y°), from which the thesis follows. □

Lemma L.9. *Given a nonempty regular set X in R^n:*
diam(X°) = *diam*(X) = *diam*($[X]$).

Proof. From L.2 and from the definition of diameter, diam(X°) $\leq$ diam(X) $\leq$ diam($[X]$). For the other direction, suppose (by contradiction) that diam($[X]$) $>$ diam(X°), i.e., s = sup{dist(x, y)|x,$y \in [X]$} $>$ sup{dist(x, y)|x,$y \in X^\circ$} = s'. Then, there exists $d > 0$ such that $s - d = s'$. Since for all $\varepsilon > 0$, there exist x,$y \in [X]$ such that dist(x, y) $> s - \varepsilon$, fix $\varepsilon = d/3$, then dist(x, y) $> s'$ and so ball(x, ε) $\cap X^\circ = \emptyset$ or ball(y, ε) $\cap X^\circ = \emptyset$. From this result, $x \notin [X]$ or $y \notin [X]$, a contradiction. Thus, diam($[X]$) $\leq$ diam(X°) from which the thesis follows. □

Lemma L.10. *Given 2 nonempty regular sets X and Y in* R^n*:*
diam($X \cup Y$) $\leq$ *dist*(X, Y) + *diam*(X) + *diam*(Y).

Proof. This result follows easily from the triangular inequality. □

Lemma L.11. *Given 2 open regular sets X and Y in R^n. If X and Y have finite diameter, then $X \cap Y$ and $[X \cup Y]^\circ$ are open regular sets with finite diameters.*

Proof. From L.7, $X \cap Y$ and $[X \cup Y]^\circ$ are open regular sets. In the first case, the condition on the diameter follows from $(X \cap Y) \subseteq X$ while in the second case from L.8, L.9, and L.10. □

Lemma L.12. *Given 2 nonempty regular sets X,$Y \subseteq$* R^n *with at least 1 of finite diameter:*
$\exists x$,y($x \in \partial(X) \wedge y \in \partial(Y) \wedge$ *dist*(x, y) = *dist*(X, Y)).

Proof. First, we show $\exists x$,y($x \in [X] \wedge y \in [Y] \wedge$ dist(x, y) = dist(X, Y)).

Assume that both X and Y have finite diameter. By definition of dist(X, Y), one can find a sequence (a_i) in X and a sequence (b_i) in Y such that $\lim_{i\to\infty}$ dist(a_i, b_i) = dist(X, Y). Since diam(X), diam(Y) $< +\infty$, there exist a, $b \in \mathrm{R}^n$ and subsequences (a_j) and (b_j) of (a_i) and (b_i), respectively, such that $a = \lim_{j\to\infty} a_j$ and $b = \lim_{j\to\infty} b_j$. Clearly, $a \in [X]$, $b \in [Y]$, and dist(a, b) = $\lim_{j\to\infty}$ dist(a_j,b_j) = dist(X, Y).

If diam(Y) = $+\infty$, we proceed as before to isolate $a \in [X]$, since $[X]$ is compact. Then, we consider sequence (b_i) in Y such that $\lim_{i\to\infty}$ dist(a, b_i) = dist(X, Y). For any positive r, (b_i) $\cap$ {$y \in [Y]$ | dist(y, a) $<$ dist(X, Y) + r} contains an infinite subsequence of (b_i), call it (c_i). Then, there exists $c = \lim_{j\to\infty} c_j$ for some subsequence (c_j) of (c_i). Clearly, $c \in [Y]$ and dist(a,c) = dist(X, Y).

Now, assume that we isolated x and y satisfying $\exists x$,y($x \in [X] \wedge y \in [Y] \wedge$ dist(x, y) = dist(X, Y)). It is easy to see that if $x \in X^\circ$, in R^n one can find $x' \in [X]$ such that dist(x', y) $<$ dist(x, y). Thus, $x \in \partial(X)$. Similarly for y. □

Lemma L.13. *Given a nonempty regular set X in* R^n :
$WConx(X) \to \forall d(0 \leq d < diam(X) \to \exists x,y(x,y \in X \wedge dist(x, y) = d))$.
If X has finite diameter:
$\exists x,y(x,y \in [X] \wedge dist(x, y) = diam(X))$.

Proof. The result is trivial for $d = 0$. For $d > 0$, let X be open (for X closed, it follows from this). Fix $d \in (0, \mathrm{diam}(X))$ and fix $x,z \in X$ such that $\mathrm{dist}(x, z) = d'$ with $d < d' <$ $\mathrm{diam}(X)$ (these points exist from the definition of diam). Let $A = \{y \in X | \mathrm{dist}(x, y) \leq d\}$ and $B = \{y \in X \mid \mathrm{dist}(x, y) > d\}$. Clearly, $X = A \cup B$ and both A and B are nonempty since $x \in A$ and $z \in B$. If there exists $u \in A$ such that $\mathrm{dist}(x, u) = d$, we are done. Otherwise, we have that A is open in R^n since A is open in X and X is open in R^n. Also, note that B is open since, from the definition, B is open in X. From WConx(X), there exists $\upsilon \in [A] \cap [B]$. Thus, $\mathrm{dist}(x, \upsilon) = d$.

Let $O_x = \mathrm{ball}(x, \delta)$, for some δ, such that $[O_x] \subset X$ and fix $y \in B$ such that $\mathrm{dist}(y, \upsilon) < \delta/2$ (this point exists since $\upsilon \in \partial(B)$). Fix the line through y and x, call it L. Let $\{x_1, x_2\} = \partial(L \cap [O_x])$ with $d_1 = \mathrm{dist}(y, x_1) < d_2 = \mathrm{dist}(y, x_2)$. Then, $d_1 < d < d_2$. Since $[y, x_2] \subset L$ is compact, the function dist on $[y, x_2] \times \{y\}$ assumes all values in $[0, d_2]$. Since dist, on $[y, x_1] \times \{y\}$, assumes all values in $[0, d_1]$ and it is strictly increasing, there exists $x^* \in [x_1, x_2] \subset (L \cap [O_x])$ such that $\mathrm{dist}(x^*, y) = d$. Since $x^*, y \in X$, we are done.

For the second claim. By definition of diam(X), one can find 2 sequences (a_i) and (b_i) in X such that $\lim_{i\to\infty} \mathrm{dist}(a_i, b_i) = \mathrm{diam}(X)$. Since $\mathrm{diam}(X) < +\infty$, there exist $a,b \in$ R^n and subsequences (a_j) and (b_j) of (a_i) and (b_i), respectively, such that $a = \lim_{j\to\infty} a_j$ and $b = \lim_{j\to\infty} b_j$. Clearly, $a \in [X]$, $b \in [X]$, and $\mathrm{dist}(a, b) = \lim_{j\to\infty} \mathrm{dist}(a_j, b_j) =$ $\mathrm{diam}(X)$. □

Lemma L.14. *Given 2 nonempty regular sets* $X, Y \subseteq \mathrm{R}^n$ *with finite diameter*:
$(WConx(X) \wedge WConx(Y) \wedge dist(X, Y) = 0) \to WConx(X \cup Y)$.

Proof. If not, then there exist A° and B° nonempty such that $(X \cup Y)^\circ = A^\circ \cup B^\circ$ and $[A] \cap [B] = \emptyset$. If $A^\circ \cap X^\circ \neq \emptyset$ and $B^\circ \cap X^\circ \neq \emptyset$, then $A^\circ \cap X^\circ$ and $B^\circ \cap X^\circ$ contradict WConx(X). Thus, either $X^\circ \subseteq A^\circ$ or $X^\circ \subseteq B^\circ$. Similarly for Y. Thus, we have $X^\circ = A^\circ$ or $X^\circ = B^\circ$ and $Y^\circ = A^\circ$ or $Y^\circ = B^\circ$.

Among the 2 cases (1) $X^\circ = Y^\circ = A^\circ$ (or $X^\circ = Y^\circ = B^\circ$) and (2) $X^\circ = A^\circ$ and $Y^\circ =$ B° (or vice versa), the first contradicts the assumption that both A° and B° are nonempty. For case (2), assume $X^\circ = A^\circ$ and $Y^\circ = B^\circ$ (the other option is similar). From $\mathrm{dist}(X, Y) = 0$ and L.8, we have $\mathrm{dist}(A, B) = 0$. From L.1, for any regular set Z, if WConx(Z), then WWConx(Z). In particular, WWConx(X) and WWConx(Y). Our argument above applies to all possible pairs of nonempty sets A° and B° such that $(X \cup Y)^\circ = A^\circ \cup B^\circ$. This proves that WWConx($X \cup Y$) holds. Since L.10 implies that diam($X \cup Y$) is finite, it suffices to recall L.1 to conclude WConx($X \cup Y$). □

Lemma L.15. *Given 2 nonempty regular sets* $X, Y \subseteq \mathrm{R}^n$ *with finite diameter:*[22]
$[X] \cap [Y] \neq \emptyset \leftrightarrow dist(X, Y) = 0$.

Proof. From L.12 and the definition of dist. □

[22] Note that this result provides an indirect proof of $(\mathrm{WWConx}(X) \wedge \mathrm{diam}(X) < +\infty) \to$ WConx(X).

Lemma L.16. *Given an open regular set X in* R^n, *Conv*(X) $\rightarrow$*Conx*(X).

Proof. If X is empty, there is nothing to prove. If X is nonempty, we proceed by contradiction. Let $X^\circ = A^\circ \cup B^\circ$ with A°, B° both nonempty and $A \cap B = \emptyset$. Since X is convex, for any $a \in A$ and $b \in B$ the segment (a, b) should be contained in $A \cup B$. But $(a, b) \cap A$ and $(a, b) \cap B$ are open nonoverlapping segments, and an open segment cannot be split into 2 open nonoverlapping subsegments. This implies that $(a, b) \not\subset X$ and contradicts the assumption *Conv*(X). □

Lemma L.17. *Given a regular connected (Conx) region X in* R^n *and a point $p \in [X]$, there exists $Y \subseteq X$ such that Conx(Y)*, $\mathrm{diam}(Y) < +\infty$, *and* $p \in [Y]$.

Proof. If X is with finite diameter, there is nothing to show. Let $\mathrm{diam}(X) = +\infty$ and X open (if X is closed, consider X°). If $p \in X$, there exists $C_r = \mathrm{ball}(p, r) \subseteq X$ for some r, and we are done. If $p \in \partial(X)$, let $C_r = \mathrm{ball}(p, r)$ and consider $Z_r = C_r \cap X$. We show that, for some r, Z_r is connected. By contradiction, let us assume that, for all r, Z_r is not connected. Since Z_1 is open, given a point $q \in Z_1$, it is possible to find a (connected) neighborhood Q of q contained in Z_1. Fix q and let A_1 be the maximal connected part of Z_1 that contains q. Build 2 sequences of regions $A_1, A_2, \ldots$ and $B_1, B_2, \ldots,$ such that A_n is the maximal connected open region part of $Z_n = \mathrm{ball}(p, n)$, $A_n \subseteq A_{n+1}$, and $B_n = Z_n - A_n$. Since Z_n is not connected by hypothesis and A_n is a maximal connected region, we have that A_n and B_n are open, $Z_n = A_n \cup B_n$, and $A_n \cap B_n = \emptyset$. Since for $n \rightarrow +\infty$, $Z_n = X$, we have that $X = (\cup A_n) \cup (\cup B_n)$. Since, for each n, A_n and B_n are open and $A_n \cap B_n = \emptyset$, we get $(\cup A_n) \cap (\cup B_n) = \emptyset$, i.e., X is not connected, a contradiction. □

Lemma L.18. *Given a regular set X in* R^n*:*

$$WWConx(X) \rightarrow \forall d (0 \leq d < diam(X) \rightarrow \exists x,y(x,y \in X \wedge dist(x, y) = d)).$$

Proof. If WConx(X), then this claim reduces to L.13. If X is finite, then it follows from L.1 and L.13. Now, assume that X is infinite, WWConx(X) and not WConx(X). We prove something even stronger, that is, we show that each regular set X with infinite diameter contains a regular set $Y \subseteq X$ whose diameter is also infinite and such that WConx(Y). From this, applying L.13 to Y, we conclude.

Fix a rational[23] $y_0 \in X^\circ$ and let $Z_0 = \cup C$, where $C \subseteq X$, WConx(C) and $y_0 \in C^\circ$. First, since y_0 is internal, there exists $D \subseteq X$ open ball centered in y_0, and from L.16, WConx(D), thus $Z_0 \neq \emptyset$. We show that WConx(Z_0). Suppose WConx(Z_0) fails, then there exist A° and B° nonempty such that $Z_0^\circ = A^\circ \cup B^\circ$ and $[A] \cap [B] = \emptyset$. Since WConx(C) and $C^\circ \subseteq Z_0^\circ = A^\circ \cup B^\circ$, for all C in $\cup C$, we have either $A^\circ \cap C^\circ = \emptyset$ or $B^\circ \cap C^\circ = \emptyset$. Then, $C^\circ \subseteq A^\circ$ or $C^\circ \subseteq B^\circ$. However, $y_0 \in C$ for all such C, thus either for all C, $C^\circ \subseteq A^\circ$ or for all C, $C^\circ \subseteq B^\circ$. That is, either $A^\circ = \emptyset$ or $B^\circ = \emptyset$, a contradiction. This proves WConx(Z_0). Note that Z_0 is closed in X. Indeed, if $x \in X \cap \partial(Z_0)$, then WConx$(Z_0^\circ \cup \{x\})$ holds since the only hope to find a counterexample is by splitting $(Z_0^\circ \cup \{x\})^\circ$ into Z_0° and $\{x\}^\circ$, but the latter set is empty. Thus, $Z_0^\circ \cup \{x\}$ itself is one of the Cs considered in the construction of Z_0. This implies $x \in Z_0$ and so Z_0 is closed in X.

If $\mathrm{diam}(Z_0) = +\infty$, we are done. Otherwise, consider $X - Z_0$ and let y_1 be a rational in $(X - Z_0)^\circ$. We repeat the construction above to find Z_1 maximal in $X - Z_0$ such that WConx(Z_1). If $\mathrm{diam}(Z_1) = +\infty$, we are done. Otherwise, consider $X - (Z_0 \cup Z_1)$ and let

[23] A point in R^n is rational if all its coordinates are rational numbers.

y_2 be a rational point in set $(X - (Z_0 \cup Z_1))^\circ$. Again, we repeat the construction above to find Z_2 maximal in $X - (Z_0 \cup Z_1)$ such that WConx(Z_2). We proceed in this way till we find a set Z_i with WConx(Z_i) and diam(Z_i) = $+\infty$ or till we cover all of X°. In the first case, we are done because of L.13. We show that the latter case cannot happen. Indeed, in the latter case we obtain a sequence of countably many regular sets Z_i (covering X° since the rationals are dense in R^n) such that, for all i, WConx(Z_i) and diam(Z_i) $< +\infty$. Fix Z_0. If dist(Z_0, Z_i) $\neq 0$ for all $Z_i \neq Z_0$, it suffices to put $Z_0 = A$ and $\cup_{i>0} Z_i = B$ to find a contradiction to WWConx(X). Let Z_{i1} be such that dist(Z_0, Z_{i1}) = 0 and let $Z_0 \cup Z_{i1} = U_1$. By L.14, WConx(U_1). As before, there exists Z_{i2} such that dist(U_1, Z_{i2}) = 0. Let $U_1 \cup Z_{i2} = U_2$. By L.14, WConx(U_2). We proceed in this way constructing sets $U_r = U_{r-1} \cup Z_{i_r}$. Let $U_\infty = \cup U_i$. First, note that if there exists i such that $Z_i \not\subset U_\infty$, then we can reapply the argument above to get a contradiction to WWConx(X). Thus, $U_\infty = \cup Z_i$. By construction, we have WConx(U_∞). However, $U_\infty^\circ = \cup Z_i^\circ = X^\circ$, and since each Z_i is closed in X, we actually have $U_\infty = \cup Z_i = X$. This contradicts the assumption that WConx(X) fails. Thus, either that assumption is wrong (i.e., WConx(X) holds) or for some i, WConx(Z_i) and diam(Z_i) = $+\infty$, which is what we needed to prove. □

Lemma L.19. *Given a set X in* R^n*, if Y is the smallest convex set containing X (the convex hull of X), then diam*(X) =*diam*(Y).

Proof. If X is empty, there is nothing to prove. If X is nonempty, we proceed by contradiction. Since $X \subseteq Y$, we need to consider only case: diam(X) $<$ diam(Y). From this assumption, there exist $x \in Y - X$ and $y \in Y$ such that dist(x,y) $>$ diam(X). By the definition of Y, since $x \in Y - X$, there exist a,$b \in X$ such that x is between a and b. Consider the ball B of radius r = dist(x,y) centered at y and the line l through a, x, and b. Since $x \in l$, there are only 2 cases to consider: l is tangent to B or l intersects B. In the first case, dist(a,y) $> r$ = dist(x,y), contradicting the assumption. In the latter case, by the so-called Pasch axiom of Euclidean geometry, at least one between a and b has to lie outside C. Thus, we reached a contradiction again. □

9. Appendix B.

9.1. Proof of Proposition 1. **(DP)** $\mathsf{P}^*(x,y) =_{\text{def}} \forall w(\mathsf{C}(x, w) \to \mathsf{C}(y, w))$ and $[\![\mathsf{P}*(x,y)]\!]_{\alpha-\delta} = X \subseteq Y$.

Proof. We have to prove that in all the domains,
$X \subseteq Y \Leftrightarrow \forall W([W] \cap [X] \neq \emptyset \to [W] \cap [Y] \neq \emptyset)$.

($\Rightarrow$) Trivial.

($\Leftarrow$) (By contradiction) Assume $\neg(X \subseteq Y)$. Since X and Y are both open, then $\neg(X \subseteq [Y])$ by L.2 and L.5. It suffices to fix an open ball W contained in $X - [Y]$. Such a W exists because $X - [Y]$ is open and nonempty. □

(DPP) $\mathsf{PP}(x,y) =_{\text{def}} \mathsf{P}(x, y) \wedge \neg\mathrm{P}(y, x)$ and $[\![\mathsf{PP}(x,y)]\!]_{\alpha-\delta} = X \subset Y$.

Proof. This follows from the obvious equivalence: $(X \subseteq Y \wedge \neg(Y \subseteq X)) \Longleftrightarrow X \subset Y$. □

(DO) $\mathsf{O}(x,y) =_{\text{def}} \exists z(\mathsf{P}(z, x) \wedge \mathsf{P}(z, y))$ and $[\![\mathsf{O}(x,y)]\!]_{\alpha-\delta} = X \cap Y \neq \emptyset$.

Proof. We have to prove that in all the domains:
$\exists Z(Z \subseteq X \wedge Z \subseteq Y) \Leftrightarrow X \cap Y \neq \emptyset$.
($\Rightarrow$) From the hypothesis $Z \subseteq (X \cap Y)$. Since Z is nonempty in all domains, one has $X \cap Y \neq \emptyset$.
($\Leftarrow$) It suffices to fix Z open ball contained in $A = X \cap Y$. (Such a ball exists because, by L.7, A is an open set of R^n.) □

(DPO) $\mathsf{PO}(x,y) =_{\mathrm{def}} \mathsf{O}(x, y) \wedge \neg\mathsf{P}(x, y) \wedge \neg\, \mathsf{P}(y, x)$ and
$[\![\mathsf{PO}(x,y)]\!]_{\alpha-\delta} = X \cap Y \neq \emptyset \wedge \neg(X \subseteq Y) \wedge \neg(Y \subseteq X)$.

Proof. Trivial. □

(D+) $\mathsf{SUM}(z, x, y) =_{\mathrm{def}} \forall w(\mathsf{O}(w, z) \leftrightarrow (\mathsf{O}(w, x) \vee \mathsf{O}(w, y)))$ and $[\![\mathsf{SUM}(z, x, y)]\!]_{\alpha-\delta}$ $= (Z = [X \cup Y]^\circ)$.

Proof. We have to prove that in all the domains,

$$\forall W(W \cap Z \neq \emptyset \leftrightarrow (W \cap X \neq \emptyset \vee W \cap Y \neq \emptyset)) \Leftrightarrow Z = [X \cup Y]^\circ.$$

($\Rightarrow$) Since for all nonempty $W \subseteq X$ we have $W \cap X \neq \emptyset$, then $W \cap Z \neq \emptyset$. Thus, $X \subseteq Z$. Analogously, we conclude $Y \subseteq Z$. Thus, $X \cup Y \subseteq Z$. But Z is a regular open region, then $[X \cup Y]^\circ \subseteq Z$. If $W \cap Z = \emptyset$, then $W \cap X = \emptyset$ and $W \cap Y = \emptyset$. Thus, Z is contained in the smallest regular open region that contains both X and Y, i.e., $[X \cup Y]^\circ$.
($\Leftarrow$) Substitute $[X \cup Y]^\circ$ for Z in the left-hand side. Then,
($\rightarrow$) From L.2, $W\cap\ [X \cup Y]^\circ \neq \emptyset$ implies that $W\cap\ [X \cup Y] \neq \emptyset$. From L.3, $[X\cup\ Y] = X] \cup [Y]$ and then $W\cap\ [X] \neq \emptyset$ or $W\cap\ [Y] \neq \emptyset$. Since W, X, Y are open and regular, we have $W \cap X \neq \emptyset$ or $W \cap Y \neq \emptyset$.
($\leftarrow$) $X \cup Y$ is open and then $(X \cup Y) \subseteq [X \cup Y]^\circ$ (from the definition of the interior operator). Since $(W \cap X \neq \emptyset \vee W \cap Y \neq \emptyset) \leftrightarrow W\cap (X \cup Y) \neq \emptyset$ and $(X \cup Y) \subseteq [X \cup Y]^\circ$, then $W\cap\ [X\cup\ Y]^\circ \neq \emptyset$. □

(D−) $\mathsf{DIF}(z, x, y) =_{\mathrm{def}} \forall w(\mathsf{P}(w, z) \leftrightarrow (\mathsf{P}(w, x) \wedge\ \neg\ \mathsf{O}(w, y)))$ and $[\![\mathsf{DIF}(z, x, y)]\!]_{\alpha-\delta} =$ $(Z = X-\ [Y])$;

Proof. We have to prove that in all the domains,

$$\forall W(W \subseteq Z \leftrightarrow (W \subseteq X \wedge W\cap\ Y = \emptyset)) \Leftrightarrow Z = X-\ [Y].$$

($\Rightarrow$) Let $W = X-\ [Y]$. Then, $W \subseteq X$ and $W \cap Y = \emptyset$. Thus, $X - [Y] \subseteq Z$. For the other inclusion, let $W \cap (X-\ [Y]) = \emptyset$. Then, $W \subseteq X$ or $W \cap Y = \emptyset$ fails. Thus, $W \subseteq Z$ fails as well. Since this happens for all W satisfying this property, we conclude $Z \subseteq X-$ $[Y]$.
($\Leftarrow$) Substitute $X-\ [Y]$ for Z in the left-hand side. Since W, X, Y are all open and regular, $W \cap Y = \emptyset$ iff $W \cap [Y] = \emptyset$. The equivalence follows. □

(DIP) $\mathsf{IP}(x,y) =_{\mathrm{def}} \mathsf{P}(x, y) \wedge \forall z(\mathsf{C}(z, x) \rightarrow \mathsf{O}(z, y))$ and $[\![\mathsf{IP}(x,y)]\!]_{\alpha-\delta} = [X] \subseteq Y$.

Proof. We have to prove that in all the domains,
$(X \subseteq Y \wedge \forall Z([Z] \cap [X] \neq \emptyset \rightarrow Z \cap Y \neq \emptyset)) \Longleftrightarrow [X] \subseteq Y$.

($\Rightarrow$) (By contradiction) Suppose $\neg([X] \subseteq Y)$. If $\neg(X \subseteq Y)$, we are done. Otherwise, note that $Y \neq \mathrm{R}^n$. From L.7, $A = \sim[Y]$ is a regular nonempty open set. From $\neg([X] \subseteq Y)$

and $X \subseteq Y$, there exists a point $x \in \partial(X)$ and $x \notin Y$. Since $x \notin Y$, $x \in [A]$. Fix a positive number r and let $Z = \text{Ball}(x,r) \cap A$. Then, $Z \cap Y = \emptyset$. From L.7, Z is regular open. Also, Z is nonempty since $x \in \partial(X)$, $x \notin Y$, and Y is regular. From L.4, $x \in [Z]$. Then, $Z \cap Y = \emptyset$ and $[Z] \cap [X] \neq \emptyset$, a contradiction.

($\Leftarrow$) From L.2, $[X] \subseteq Y$ implies $X \subseteq Y$. Now suppose $[Z] \cap [X] \neq \emptyset$ and $[X] \subseteq Y$, then we have $[Z] \cap Y \neq \emptyset$, i.e., $\exists p(p \in [Z] \wedge p \in Y)$. If $p \in Z$, then $Z \cap Y \neq \emptyset$. If $p \in \partial(Z)$, then there exists a neighborhood of p, say $I(p)$, such that $I(p) \subseteq Y$ and $I(p) \cap Z \neq \emptyset$. This means that there exists a point $p' \in I(p) \cap Z$ and so $Z \cap Y \neq \emptyset$. □

(DTPP) $\mathsf{TPP}(x, y) =_{\text{def}} \mathsf{PP}(x, y) \wedge \exists z(\text{EC}(z,x) \wedge \mathsf{EC}(z,y))$ and $[\![\mathsf{PP}(x,y)]\!]_{\alpha-\delta} = X \subset Y \wedge \partial(X) \cap \partial(Y) \neq \emptyset$.

Proof. We have to prove that in all the domains,
$X \subset Y \wedge \partial(X) \cap \partial(Y) \neq \emptyset \Leftrightarrow X \subset Y \wedge \exists Z([Z] \cap [X] \neq \emptyset \wedge Z \cap X = \emptyset \wedge [Z] \cap [Y] \neq \emptyset \wedge Z \cap Y = \emptyset)$.

($\Rightarrow$) Let $p \in \{\partial(X) \cap \partial(Y)\}$. The thesis follows considering an open ball Z such that $p \in [Z]$, and $Z \cap Y = \emptyset$.

($\Leftarrow$) From the hypothesis, we have that $\partial(Z) \cap \partial(X) \neq \emptyset$ and $\partial(Z) \cap \partial(Y) \neq \emptyset$. From L.5 and $X \subset Y$, it follows that $[X] \subset [Y]$, therefore $\partial(X) \cap \partial(Y) \neq \emptyset$. □

(DSC) $\mathsf{SC}(x) =_{\text{def}} \forall y,z(\mathsf{SUM}(x, y, z) \rightarrow \text{C}(y, z))$ and $[\![\mathsf{SC}(x)]\!]_{\alpha-\delta} = \text{WConx}(X)$.

Proof. From the definition of WConx and considering the given domains, WConx(X) stands for $\forall Y,Z(X = Y \cup Z \rightarrow [Y] \cap [Z] \neq \emptyset)$, thus we prove that

$$\forall Y,Z(X = [Y \cup Z]^{\circ} \rightarrow [Y] \cap [Z] \neq \emptyset) \Leftrightarrow \forall Y',Z'(X = Y' \cup Z' \rightarrow [Y'] \cap [Z'] \neq \emptyset).$$

From the fact that X is a nonempty open regular set, we have $Y' \cup Z' = [Y' \cup Z']^{\circ}$. The equivalence follows considering $Y = Y'$ and $Z = Z'$. □

(DEC) $\mathsf{EC}(x, y) =_{\text{def}} \mathsf{C}(x, y) \wedge \neg\mathsf{O}(x,y)$ and $[\![\mathsf{EC}(x, y)]\!]_{\alpha-\delta} = [X] \cap [Y] \neq \emptyset \wedge X \cap Y = \emptyset$.

Proof. Trivial. □

9.2. Proof of Proposition 2. **(DC1)** $\mathsf{C}_1(x,y) =_{\text{def}} \exists z(\mathsf{S}(z) \wedge \forall z'(\mathsf{CNC}(z', z) \rightarrow (\mathsf{O}(z', x) \wedge \mathsf{O}(z', y))))$ and
$[\![\mathsf{C}_1(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{C}(x, y)]\!] = [X] \cap [Y] \neq \emptyset$.

Proof. We have to prove that in all the domains,
$[X] \cap [Y] \neq \emptyset \Leftrightarrow \exists Z,c,r(Z = \text{ball}(c, r) \wedge \forall Z',r'(Z' = \text{ball}(c, r') \rightarrow (Z' \cap X \neq \emptyset \wedge Z' \cap Y \neq \emptyset)))$.

($\Rightarrow$) It is sufficient to consider $c \in [X] \cap [Y]$ and $r > 0$.

($\Leftarrow$) (By contradiction) Suppose $[X] \cap [Y] = \emptyset$ and $Z = \text{ball}(c, r)$. If $c \notin \partial(X) \cup \partial(Y)$, then it is sufficient to consider r' small enough such that $Z' = \text{ball}(c, r') \subseteq X$ (if $c \in X$) or $Z' = \text{ball}(c, r') \subseteq Y$ (if $c \in Y$) or $Z' = \text{ball}(c, r') \subseteq \mathrm{R}^n - ([X] \cup [Y])$ (if $c \notin [X] \cup [Y]$). If $c \in \partial(X)$ and $c \notin [Y]$, i.e., $\text{dist}(c, Y) > 0$, it suffices to consider r' small enough such that $Z' = \text{ball}(c, r') \cap [Y] = \emptyset$. Analogously for $c \in \partial(Y)$, a contradiction. □

(DSR1) $\mathsf{SR}_1(x) =_{\text{def}} \forall y,z(\mathsf{SUM}(x, y, z) \rightarrow \exists s(\mathsf{S}(s) \wedge \mathsf{O}(s, y) \wedge \mathsf{O}(s, z) \wedge \mathsf{P}(s, x)))$ and $[\![\mathsf{SR}_1(x)]\!]_{\alpha-\delta} = [\![\mathsf{SR}(x)]\!] = \text{Conx}(X)$.

Proof. We have to prove that in all the domains,

$\mathrm{Conx}(X) \Leftrightarrow \forall Y,Z(X = [Y \cup Z]^{\circ} \rightarrow \exists S,c,r(S = \mathrm{ball}(c, r) \wedge S \cap Y \neq \emptyset \wedge S \cap Z \neq \emptyset \wedge S \subseteq X))$.

($\Rightarrow$) Since Y and Z are regular open, from L.3 and L.4 we have $X = [Y \cup Z]^{\circ} \supseteq Y \cup Z$. From the definition of Conx, if $X = [Y \cup Z]^{\circ} = Y \cup Z$, then $Y \cap Z \neq \emptyset$ and, by L.7, $Y \cap Z$ is open. Then, there exists a ball S such that $S \subseteq Y \cap Z$. If $X = [Y \cup Z]^{\circ} \supset Y \cup Z$, then there exists x such that $x \in \partial(Y) \cap \partial(Z)$ and $x \in X$. X is open; thus, there exists a ball S centered at x such that $S \subseteq X$, $S \cap Y \neq \emptyset$, and $S \cap Z \neq \emptyset$.

($\Leftarrow$) (By contradiction) Assume that there exist Y, Z such that $X = Y \cup Z$ and $Y \cap Z = \emptyset$. Since X is regular, $X = [Y \cup Z]^{\circ}$. But Y, Z are open and disjoint; thus, any ball S intersecting both Y and Z intersects $\mathrm{R}^n - X$ also, then $\neg S \subseteq X$, a contradiction. □

9.3. Proof of Proposition 3. **(DCC1)** $\mathsf{CCon}_1(z, x, y) =_{\mathrm{def}} \exists z'(\mathsf{CG}(z', z) \wedge \mathsf{C}_1(z', x) \wedge \mathsf{C}_1(z', y))$ and

$[\![\mathsf{CCon}_1(z, x, y)]\!]_{\alpha} = \exists r,r',p,q(r,r' \in [Z] \wedge p \in [X] \wedge q \in [Y] \wedge \mathrm{dist}(p, q) = \mathrm{dist}(r, r'))$.

Proof. From Proposition 2 and the assumption that $[\![\mathsf{CG}(x, y)]\!] = \mathrm{Congr}(X, Y)$, we have to prove that in Φ_{α},

$\exists r,r',p,q(r,r' \in [Z] \wedge p \in [X] \wedge q \in [Y] \wedge \mathrm{dist}(p, q) = \mathrm{dist}(r, r')) \Leftrightarrow \exists Z'(\mathrm{Congr}(Z', Z) \wedge [Z'] \cap [X] \neq \emptyset \wedge [Z'] \cap [Y] \neq \emptyset)$.

($\Rightarrow$) It is sufficient to consider $Z' = f(Z)$, where f is an isometry in R^n with $f(r) = p$ and $f(r') = q$.

($\Leftarrow$) From the fact that the congruence relation preserves distance. □

9.4. Proof of Proposition 4. **(DS2)** $\mathsf{S}_2^*(x) =_{\mathrm{def}} \mathsf{SR}(x) \wedge \forall y,z((\mathsf{CG}(x, y) \wedge \mathsf{PO}(x, y) \wedge \mathsf{DIF}(z, x, y)) \rightarrow \mathsf{SR}(z))$ and $[\![\mathsf{S}_2^*(x)]\!]_{\beta} = [\![\mathsf{S}(x)]\!]$ and $[\![\mathsf{S}_2^*(x)]\!]_{\alpha,\gamma,\delta} \neq [\![(x)]\!]$.

Proof. A counterexample in D_{α} is given by a region equal to R^n minus a closed ball. This region satisfies the definition but is not a sphere in R^n. The definition does not work for D_{γ} and D_{δ} because in these domains, a region z satisfying $\mathsf{DIF}(z, x, y)$ must be already a connected region. Our direct proof that the definition above is correct in D_{β} is quite complicated, and it is not reported here. However, note that $[\![\mathsf{S}_2^*(x)]\!]_{\beta} = [\![\mathsf{S}(x)]\!]$ follows indirectly from the results in Linking Via Explicit Definitions section. □

9.5. Proof of Proposition 6. **(DC4*)** $\mathsf{C}_{4^*}(x,y) =_{\mathrm{def}} \forall z(\mathsf{CCon}(z, x, y))$ and $[\![\mathsf{C}_4^*(x,y)]\!]_{\beta,\delta} = [\![\mathsf{C}(x,y)]\!] = [X] \cap [Y] \neq$ and $[\![\mathsf{C}_4^*(x,y)]\!]_{\alpha,\gamma} \neq [\![(x,y)]\!]$.

Proof. First, we show that the following equivalence holds in Φ_{β} and Φ_{δ}:

$[X] \cap [Y] \neq \emptyset \Leftrightarrow \forall Z(\mathrm{dist}(X, Y) \leq \mathrm{diam}(Z))$.

Since we restrict ourselves to domains with finite regions, we can apply L.15 to get $[X] \cap [Y] \neq \emptyset$ iff $\mathrm{dist}(X, Y) = 0$. Note that for all $\varepsilon > 0$, there exists Z nonempty such that $\mathrm{diam}(Z) < \varepsilon$, then $\mathrm{dist}(X, Y) = 0$ iff $\forall Z\ (\mathrm{dist}(X, Y) \leq \mathrm{diam}(Z))$.

In D_{α} and D_{γ}, the interpretation $[X] \cap [Y] \neq$ fails. Consider, for example, regions $X = \{(a, b) \in \mathrm{R}^2 \mid a > 0 \text{ and } b \geq 1/a\}$ and $Y = \{(a, b) \in \mathrm{R}^2 | b \leq 0\}$; it is easy to verify that all the constraints on the domains are satisfied and that $\mathrm{dist}(X, Y) = 0$ although $[X] \cap [Y] = \emptyset$. □

9.6. Proof of Proposition 7. **(DP4*)** $\mathsf{P}_4^*(x,y) =_{\mathrm{def}} \forall z(\mathsf{C}_4^*(z, x) \rightarrow \mathsf{C}_4^*(z, y))$ and $[\![\mathsf{P}_4^*(x,y)]\!]_{\alpha-\delta} = [\![\mathsf{P}(x,y)]\!] = X \subseteq Y$.

Proof. First, note that in all the domains, the interpretation of relation $\mathsf{C}^*_4(x,y)$ as defined in (DC4*) is dist(X, Y) = 0. This follows from the proof of B.5 dropping the first step through L.15. Thus, we have to prove that in all the domains,
$X \subseteq Y \Leftrightarrow \forall Z(\text{dist}(Z, X) = 0 \rightarrow \text{dist}(Z, Y) = 0)$.

($\Rightarrow$) Trivial.
($\Leftarrow$) (By contradiction) Suppose $\neg X \subseteq Y$. Since X and Y are both open, there exists a nonempty regular open set V such that $V \subset X$ and $V \cap Y = \emptyset$. Let Z be a nonempty open ball contained in V such that $\partial(Z) \cap \partial(V) = \emptyset$. We have dist$(Z, X)$ = 0 and dist$(Z, Y) \neq 0$, a contradiction. □

(DP4$^+$) $\mathsf{P}^+_4(x,y) =_{\text{def}} \forall z,w(\mathsf{CCon}(w, z, x) \rightarrow \mathsf{CCon}(w, z, y))$ and
$[\![\mathsf{P}^+_4(x,y)]\!]_{\alpha-\delta} = [\![\mathsf{P}(x,y)]\!] = X \subseteq Y$.

Proof. It suffices to prove that in all the domains,
$X \subseteq Y \Leftrightarrow \forall Z,W(\text{dist}(Z, X) \leq \text{diam}(W) \rightarrow \text{dist}(Z, Y) \leq \text{diam}(W))$.
($\Rightarrow$) Directly from the definition of distance.
($\Leftarrow$) (By contradiction) Suppose $\neg(X \subseteq Y)$. Since X and Y are both open, there exists a nonempty finite open ball $Z \subset X$ such that dist$(Z, Y) > 0$, thus Z nontangential to $X - Y$. Clearly, dist$(Z, X) = 0$, and we can always find W such that $0 <$ diam$(W) <$ dist(Z, Y), a contradiction. □

(DCl4) $\mathsf{Closer}_4(z, x, y) =_{\text{def}} \exists a(\mathsf{CCon}(a, z, x) \wedge \neg\, \mathsf{CCon}(a,z,y))$ and
$[\![\mathsf{Closer}_4(z, x, y)]\!]_{\alpha-\delta} = [\![\mathsf{Closer}(z, x, y)]\!] = \text{dist}(Z, X) < \text{dist}(Z, Y)$.

Proof. It suffices to prove that in all the domains,
$\text{dist}(Z, X) < \text{dist}(Z, Y) \Leftrightarrow \exists A(\text{dist}(X, Z) \leq \text{diam}(A) \wedge \neg\text{dist}(Z, Y) \leq \text{diam}(A))$.

This follows from the fact that $\exists A(\text{dist}(X, Z) \leq \text{diam}(A) \wedge \neg\text{dist}(Z, Y) \leq \text{diam}(A))$ is equivalent to $\exists A(\text{dist}(X, Z) \leq \text{diam}(A) < \text{dist}(Z, Y))$ and that for all $\varepsilon > 0$, we can find a nonempty open ball A such that $0 < \text{diam}(A) < \varepsilon$. □

9.7. Proof of Proposition 8. **(DP6)** $\mathsf{P}_6(x,y) =_{\text{def}} \forall z(\mathsf{C}(z, x) \rightarrow \mathsf{C}(z, y))$ and $[\![\mathsf{P}_6(x,y)]\!]_{\alpha-\delta} = [\![\mathsf{C}(x,y)]\!] = X \subseteq Y$.

Proof. From the proof of case (DP) in Proposition 1. □

(DCM6) $\mathsf{Compl}_6(y, x) =_{\text{def}} \forall z(\mathsf{C}(z, y) \leftrightarrow \neg\, \mathsf{IP}(z, x))$ and $[\![\mathsf{Compl}_6(y, x)]\!]_{\alpha-\delta} = (Y = (\mathrm{R}^n - X)^\circ)$.

Proof. Using the results of Proposition 1, we have to prove that in all the domains,
$Y = (\mathrm{R}^n - X)^\circ \Leftrightarrow \forall Z([Z] \cap [Y] \neq \emptyset \leftrightarrow \neg([Z] \subseteq X))$.
($\Rightarrow$) If $Y = (\mathrm{R}^n - X)^\circ$, then $[Z] \cap [Y] \neq \emptyset \leftrightarrow [Z] \cap (\mathrm{R}^n - X) \neq \emptyset \leftrightarrow \neg[Z] \subseteq X$.
($\Leftarrow$) (By contradiction) Assume $Y \neq (\mathrm{R}^n - X)^\circ$. We have to prove that (i) $\exists Z([Z] \cap [Y] \neq \emptyset \wedge [Z] \subseteq X)$ or (ii) $\exists Z([Z] \cap [Y] = \emptyset \wedge \neg([Z] \subseteq X))$. If $X \cap Y \neq \emptyset$, to verify (i) it is sufficient to consider $[Z] \subseteq X \cap Y$ (Z exists since X and Y are regular open). If $X \cap Y = \emptyset$, then $Y \subset (\mathrm{R}^n - X)^\circ$ and since both these sets are open, there exists a nonempty open ball Z such that $Z \subset (\mathrm{R}^n - X)^\circ$ and $Z \cap Y = \emptyset$. Thus, taking Z small enough, we have $[Z] \subset (\mathrm{R}^n - X)^\circ - Y$. Then, condition (ii) is verified. □

(DSR6) $\mathsf{SR}_6(x) =_{\text{def}} \forall y,z,w((\mathsf{SUM}(x,y,z) \wedge \mathsf{Compl}_6(w, x)) \rightarrow \exists v(\mathsf{SC}(v) \wedge \mathsf{O}(v, y) \wedge \mathsf{O}(v, z) \wedge \neg\, \mathsf{C}(v, w)))$.
$[\![\mathsf{SR}_6(x)]\!]_{\alpha-\delta} = [\![\mathsf{SR}(x)]\!] = \text{Conx}(X)$.

Proof. (Recall definition (DSC) from Proposition 1.) We have to prove that in all the domains,

$\mathrm{Conx}(X) \Leftrightarrow \forall Y,Z,W\ (X = [Y \cup Z]^\circ \wedge W = (\mathrm{R}^n - X)^\circ) \rightarrow \exists V(\mathrm{WConx}(V) \wedge V \cap Y \neq \emptyset \wedge V \cap Z \neq \emptyset \wedge [V] \cap [W] = \emptyset)$.

First, since X is open, we have $[V] \cap [W] = \emptyset \leftrightarrow [V] \cap (\mathrm{R}^n - X) = \emptyset \leftrightarrow [V] \subseteq X$.

($\Rightarrow$) Suppose $X = [Y \cup Z]^\circ$. From L.3 and L.4, since these are all open nonempty regions, we have $Y \cup Z \subseteq X$.

(a) If $Y \cup Z = X$, then (from the definition of $\mathrm{Conx}(X)$ and L.7) set $A = Y \cap Z$ is open, nonempty, and regular. The thesis follows since $A \subseteq X$ and we can always find a nonempty open ball V in A (so that $\mathrm{WConx}(V)$ by L.16, the definition of WConx and L.2) such that $[V] \subseteq A$.

(b) If $X = [Y \cup Z]^\circ$ and $Y \cup Z \subset X$, then $\partial(Y) \cap \partial(Z) \cap X \neq \emptyset$. Since X is open, fix a nonempty open ball V centered in $p \in \partial(Y) \cap \partial(Z) \cap X$ with $[V] \subseteq X$. From the definition of boundary, we have that $V \cap Y \neq \emptyset \wedge V \cap Z \neq \emptyset$.

($\Leftarrow$) (By contradiction) Suppose there exist nonempty open regions Y and Z such that $X = Y \cup Z$ and $Y \cap Z = \emptyset$. From the regularity of X, $X = Y \cup Z = [Y \cup Z]^\circ$. Fix V such that $\mathrm{WConx}(V) \wedge V \cap Y \neq \emptyset \wedge V \cap Z \neq \emptyset$, we prove that $\neg([V] \subseteq X)$. Let us consider $VY = V \cap Y$, $VZ = V \cap Z$, and $U = VY \cup VZ$ (note that VY and VZ not necessarily belong to a domain D among $D_{\alpha-\delta}$, but this does not invalidate our inference because these regions are not used as values of some variable). Clearly, U is in V.

(a) If $U \subset V$, then $\exists p(p \in V \wedge p \notin VY \wedge p \notin VZ)$. But $X = Y \cup Z$ and so $p \notin X$. From this, $V \subseteq X$ is false, and by L.2, it follows that $\neg([V] \subseteq X)$.

(b) If $U = V$, from the definition of $\mathrm{WConx}(V)$, it follows that $[VY] \cap [VZ] \neq \emptyset$. But from $Y \cap Z = \emptyset$, we get $VY \subseteq Y$ and $VZ \subseteq Z$ so that $VY \cap VZ = \emptyset$. This means that $\exists p \in \partial(VY) \cap \partial(VZ)$. Given a neighborhood $I(p)$ of p, we have $I(p) \cap VY \neq \emptyset$ and $I(p) \cap VZ \neq \emptyset$ and so $I(p) \cap Y \neq \emptyset$ and $I(p) \cap Z \neq \emptyset$. Thus, $p \in \partial(Y) \cap \partial(Z)$. Since Y and Z are open, $p \notin Y \cup Z$ and, in particular, $p \notin X$. Finally, from $(VY \cup VZ) = V$ and L.3, it follows that $p \in [V]$ and $p \notin X$, i.e., $\neg[V] \subseteq X$, a contradiction. □

9.8. Proof of Proposition 10. **(DC3*)** $\mathsf{C}_3^*(x,y) =_{\mathrm{def}} \forall z(\mathsf{Conj}(z, z, x, y))$ and $[\![\mathsf{C}_3^*(x,y)]\!]_{\alpha-\delta} = (\mathrm{dist}(X, Y) = 0)$.

Proof. We have to prove that in all the domains,

$\mathrm{dist}(X, Y) = 0 \Leftrightarrow \forall Z \exists z,z',x,y(z, z' \in [Z] \wedge x \in [X] \wedge y \in [Y] \wedge \mathrm{dist}(z, z') = \mathrm{dist}(x, y))$.

($\Rightarrow$) If $\mathrm{dist}(X, Y) = 0$, then for each $d > 0$, there exist $x \in X$ and $y \in Y$ such that $\mathrm{dist}(x, y) < d$. In particular, take a nonempty open ball $B \subseteq Z$ and consider $x \in X$ and $y \in Y$ such that $\mathrm{dist}(x, y) < \mathrm{diam}(B)$. From L.13 and L.16, there exist $z, z' \in B$ such that $\mathrm{dist}(z, z') = \mathrm{dist}(x, y)$.

($\Leftarrow$) (By contradiction) Let $\mathrm{dist}(X, Y) = d > 0$. It is sufficient to consider Z such that $\mathrm{diam}(Z) < d$ to get a contradiction. □

(DP3) $\mathsf{P}_3(x,y) =_{\mathrm{def}} \forall z(\mathsf{C}_3^*(z, x) \rightarrow \mathsf{C}_3^*(z, y))$ and $[\![\mathsf{P}_3(x,y)]\!]_{\alpha-\delta} = [\![\mathsf{P}(x,y)]\!] = X \subseteq Y$.

Proof. This equivalence follows from above and B.6. □

9.9. Proof of Proposition 11. **(DC5*)** $\mathsf{C}^*_5(x,y) =_{\text{def}} \neg\exists z(\mathsf{Closer}(x, z, y))$ and $[\![\mathsf{C}^*_5(x,y)]\!]_{\alpha-\delta}$ = (dist$(X, Y) = 0$).

Proof. We have to prove that in all the domains,
dist$(X, Y) = 0 \Leftrightarrow \neg\exists Z(\text{dist}(X, Z) < \text{dist}(X, Y))$.

($\Rightarrow$) Trivial.
($\Leftarrow$) (By contradiction) Let dist$(X, Y) > 0$. It suffices to put $Z = X$ to reach a contradiction. □

(DP5) $\mathsf{P}_5(x, y) =_{\text{def}} \forall z\ (\mathsf{C}^*_5(z,x) \to \mathsf{C}^*_5(z,y))$ and $[\![\mathsf{P}_5(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{C}(x,y)]\!] = X \subseteq Y$.

Proof. It follows from B.6. □

(DFD5) $\mathsf{FD}_5(x) =_{\text{def}} \exists z(\forall x',x''((\mathsf{P}_5(x', x) \wedge \mathsf{P}_5(x'', x)) \to \mathsf{Closer}(x', x'', z)))$ and
$[\![\mathsf{FD}_5(x)]\!]_{\alpha-\delta}$ = diam$(X) < +\infty$.

Proof. It suffices to prove the following for all X:
diam$(X) < +\infty \Leftrightarrow \exists Z(\forall X',X''((X' \subseteq X \wedge X'' \subseteq X) \to \text{dist}(X', X'') < \text{dist}(X', Z)))$.
($\Rightarrow$) If diam$(X) < +\infty$, then consider a nonempty open ball Z such that dist$(Z, X) >$ diam(X). (Clearly, such a ball exists in R^n.)
($\Leftarrow$) (By contradiction) Assume diam$(X) = +\infty$ and let dist$(X, Z) = d$. Since diam(X) $= +\infty$, we can choose a real number r and 2 points $x,y \in X$ such that dist$(x, y) > d + 2r$ and dist$(x, Z) \le d + r$. Let $X' \subseteq X$ be a ball centered at x with diameter less than or equal to r and $X'' \subseteq X$ a ball centered at y with diameter less than or equal to r as well (these balls exist because X is open). Then, dist$(X', X'') >$ dist(X', Z), a contradiction. □

(DC5) $\mathsf{C}_5(x, y) =_{\text{def}} \exists z, w(\mathsf{FD}_5(z) \wedge \mathsf{FD}_5(w) \wedge \mathsf{P}_5(z, x) \wedge \mathsf{P}_5(w, y) \wedge \mathsf{C}^*_5(z, w))$ and
$[\![\mathsf{C}_5(x,y)]\!]_{\alpha-\delta} = [\![(x,y)]\!] = [X] \cap [Y] \neq \emptyset$.

Proof. We have to prove the following equivalence in all the domains:
$[X] \cap [Y] \neq \emptyset \Leftrightarrow \exists Z,W(\text{diam}(Z) < +\infty \wedge \text{diam}(W) < +\infty \wedge Z \subseteq X \wedge W \subseteq Y \wedge$ dist$(Z, W) = 0)$.
($\Rightarrow$) Fix $p \in [X] \cap [Y]$. In $\Phi_{\alpha,\beta}$, a region Z with finite diameter such that $Z \subseteq X$ and $p \in [Z]$ is given by ball$(p, r) \cap X$. Analogously for W. In $\Phi_{\gamma,\delta}$, by L.17, there exists $Z \subseteq X$ such that diam$(Z) < +\infty$ and $p \in [Z]$. Analogously, for $W \subseteq Y$. Since $p \in [Z] \cap [W]$, we have dist$(Z, W) = 0$.
($\Leftarrow$) From the hypothesis, we have that diam$(Z) < +\infty \wedge$ diam$(W) < +\infty \wedge$ dist$(Z, W) = 0$. From L.15, we have $[Z] \cap [W] \neq \emptyset$. But $Z \subseteq X$ and $W \subseteq Y$, thus $[X] \cap [Y] \neq \emptyset$. □

10. Appendix C.

10.1. Proof of Lemma 1. The interpretations of MSP and ΣSS are obtained by substituting the interpretations of the components in the definition. For this reason, there is nothing to prove, and in the attempt to improve readability, we simply write $[\![\mathsf{MSP}]\!](X, Y)$ and $[\![\Sigma\mathsf{SS}]\!](X)$ for the interpretations of $\mathsf{MSP}(x, y)$ and $\Sigma\mathsf{SS}(x)$, respectively. Given this premise, we prove that

(A.1) $[\![\mathsf{SCG}(x, y)]\!]_{\alpha-\delta} = \exists c_x,c_y,r\ (X = \text{Ball}(c_x, r) \wedge Y = \text{Ball}(c_y, r))$;

(A.2) $[\![\mathsf{EqD}(x, y, x', y')]\!]_{\alpha-\delta} = \exists c_x,c_y,c'_x,c'_y,r,r'(X = \text{Ball}(c_x, r) \wedge Y = \text{Ball}(c_y, r') \wedge X' = \text{Ball}(c'_x, r) \wedge Y' = \text{Ball}(c'_y, r') \wedge \neg X \subseteq Y \wedge \neg Y \subseteq X \wedge \neg X' \subseteq Y' \wedge \neg Y' \subseteq X' \wedge$ dist(c_x,c_y) = dist$(c'_x,c'_y))$;

(A.3) $[\![\mathsf{SCG}(x, y)]\!]_{\alpha-\delta}$ = Congr$(X, Y) \wedge [\![\Sigma\mathsf{SS}]\!](X) \wedge [\![\Sigma\mathsf{SS}]\!](Y)$;

(A) $[\![\mathsf{CG}_1(x, y)]\!]_{\alpha-\delta}$ = Congr(X, Y).

Proof. **(A.1)** Let X = Ball(c_x, r_x) and Y = Ball(c_y, r_y). We have to prove the following equivalence in all the domains:

Congr$(X, Y) \Leftrightarrow X = Y \vee \exists Z,W,c,r_z,r_w$ (Z = Ball$(c, r_z) \wedge$ = Ball$(c, r_w) \wedge [Z] \cap [X] \neq \emptyset \wedge Z \cap X = \emptyset \wedge [Z] \cap [Y] \neq \emptyset \wedge Z \cap Y = \emptyset \wedge X \subset W \wedge \partial(X) \cap \partial(W) \neq \emptyset \wedge Y \subset W \wedge \partial(Y) \cap \partial(W) \neq \emptyset$.

($\Rightarrow$) From $r_x = r_y$, one has diam(X) = diam(Y). If $X = Y$, we are done. Otherwise, $c_x \neq c_y$ and X, Y properly overlap or are disjoint. Fix $c \notin [X \cup Y]$ such that dist(c_x, c) = dist$(c_y, c) = r$, and let Z = Ball$(c, r - r_x)$. By construction, $[Z] \cap [X] \neq \emptyset \wedge Z \cap X = \emptyset \wedge [Z] \cap [Y] \neq \emptyset \wedge Z \cap Y = \emptyset$. Let W = Ball$(c, r + r_x)$. By construction, X,$Y \subset W$ and $\partial(X) \cap \partial(W) \neq \emptyset$, $\partial(Y) \cap \partial(W) \neq \emptyset$.

($\Leftarrow$) If $X = Y$, we are done. If not, from the hypothesis X,$Y \subset (W - Z)$, $[Z] \cap [X] \neq \emptyset$, $[Z] \cap [Y] \neq \emptyset$ and $\partial(X) \cap \partial(W) \neq \emptyset$, $\partial(Y) \cap \partial(W) \neq \emptyset$. By triangular inequality, the minimum distance between a point in $\partial(W)$ and a point in $\partial(Z)$ is $r_w - r_z$. Then, $r_w - r_z \leq$ diam(X). Analogously, $r_w - r_z \leq$ diam(Y). Assume $r_w - r_z <$ diam(X). From $Z \cap X = \emptyset$, there exists a point in X such that its distance from c is higher than r_w, a contradiction. Then, $r_w - r_z$ = diam(X). Analogously, $r_w - r_z$ = diam(Y). Finally, diam(X) = diam(Y). □

(A.2) Let X = Ball(c_x, r), Y = Ball(c_y, r'), X' = Ball(c'_x, r'), and Y' = Ball (c'_y, r') such that $\neg X \subseteq Y \wedge \neg Y \subseteq X \wedge \neg X' \subseteq Y' \wedge \neg Y' \subseteq X'$. We have to prove the following equivalence in all the domains:

dist(c_x,c_y) = dist$(c'_x,c'_y) \Leftrightarrow \exists Z,W,c_z,c_w,r''$ (Z = Ball$(c_z, r'') \wedge$ = Ball$(c_w, r'') \wedge$ Int$D(z, x, y) \wedge$ Int$D(w, x', y')$).

($\Rightarrow$) The minimum sphere S containing X and Y has diameter dist$(c_x,c_y) + r + r'$. Analogously, dist$(c'_x,c'_y) + r + r'$ is the radius of the minimal sphere S' containing X' and Y'. Thus, S and S' are congruent. Also, by triangular inequality, the center of S is between c_x,c_y, i.e., X and Y are internally diametrical to S. Analogously for S'.

($\Leftarrow$) Since Z has X and Y as internally diametrical spheres, then the center of Z is between the centers of X and Y, so r'' = dist$(c_x,c_y) + r + r'$. Analogously for W. Then, dist(c_x,c_y) = dist(c'_x,c'_y). □

(A.3) Assume $[\![\Sigma\mathsf{SS}]\!](X)$ and $[\![\Sigma\mathsf{SS}]\!](Y)$, then we have to prove the following equivalence in all the domains:

Congr$(X, Y) \Leftrightarrow \forall S([\![\mathsf{MSP}]\!](S,X) \to \exists S'([\![\mathsf{MSP}]\!]\,(S',Y) \wedge$ Congr$(S, S'))) \wedge$
$\forall S([\![\mathsf{MSP}]\!](S,Y) \to \exists S'([\![\mathsf{MSP}]\!]\,(S',X) \wedge$ Congr$(S, S'))) \wedge$
$\forall S,U,S',U([\![\mathsf{MSP}]\!](S,X) \wedge [\![\mathsf{MSP}]\!](U,X) \wedge [\![\mathsf{MSP}]\!]XS',Y) \wedge [\![\mathsf{MSP}]\!](U',Y) \wedge$
Congr$(S, S') \wedge$ Congr$(U, U')) \to [\![\mathsf{EqD}]\!](S, U, S', U')$.

($\Rightarrow$) Let f be an isometry such that $Y = f(X)$ and put $S' = f(S)$ for S in X, $S' = f^{-1}(S)$ for S in Y. The conditions follow easily.

($\Leftarrow$) We show that there exists an isometry f such that $Y = f(X)$. We write c_i for the center of sphere S_i and c'_i for the center of sphere S'_i. For each pair S_i, S_j of maximal spheres S in X, let x_{ij}, x_{ji} be the points on the boundary of S_i, S_j, respectively, such that dist(x_{ij},x_{ji}) = diam$(S_i \cup S_j)$. Let c_i,..., x_{mn},... be a list of all centers of maximal spheres in X and of all points isolated above. We show that there exists an isometry f such that $c'_i = f(c_i)$ and dist(x,y) = dist$(f(x), f(y))$ for any pair of points x,y in the list and that $S' = f(S)$ for each maximal sphere S in X (and so $S' = f^1(S)$ for S in Y). From the first 2 conditions in the hypothesis, we can choose f such that dist(x,y) = dist($f(x), f(y)$) for x,y centers of maximal spheres in X. Relation dist(x_{ij},x_{ji}) = dist($f(x_{ij}), f(x_{ji})$) can also be satisfied because of the third constraint in the hypothesis. The other pairs follow from triangular inequality. From Euclidean geometry, we know that this function can be extended to an isometry on the whole space. Since the center of a maximal sphere S is mapped to the center of its congruent sphere S' (and so the points in the boundary where f is constrained, if any) and f is an isometry, we also have $S' = f(S)$. We conclude that $Y = f(X)$. $\square$

(A) We have to prove the following equivalence in all the domains:
$\mathrm{Congr}(X, Y) \Leftrightarrow \forall Z([\![\Sigma\mathsf{SS}]\!]Z) \rightarrow \exists Z'(\mathrm{Congr}(Z, Z') \wedge$
$\forall S,S'(([\![\mathsf{MSP}]\!](S, Z) \wedge [\![\mathsf{MSP}]\!](S', Z') \wedge \mathrm{Congr}(S, S')) \rightarrow$
$((S \subseteq X \leftrightarrow S' \subseteq Y) \wedge (S \subseteq Y \leftrightarrow S' \subseteq X) \wedge$
$((S \cap X \neq \emptyset \wedge \neg S \subseteq X \wedge \neg X \subseteq S) \leftrightarrow (S' \cap Y \neq \emptyset \wedge \neg S' \subseteq Y \wedge \neg Y \subseteq S')) \wedge$
$(S' \cap X \neq \emptyset \wedge \neg S' \subseteq X \wedge \neg X \subseteq S') \leftrightarrow (S \cap Y \neq \emptyset \wedge \neg S \subseteq Y \wedge \neg Y \subseteq S))))))$

($\Rightarrow$) Let f be an isometry such that $Y = f(X)$. Fix any Z such that $[\![\Sigma\mathsf{SS}]\!]\,(Z)$ holds and let $Z' = f(Z)$. It is easy (although tedius) to verify the conditions since f is an isometry.

($\Leftarrow$) The proof splits into 4 cases. Let n be the dimension of the space.

Case (I): If X is the whole space, then so is Y. If not, it suffices to take a Z that partially overlaps Y. Analogously if Y is the whole space.

Case (II): $[\![\Sigma\mathsf{SS}]\!](X)$ and the convex hull of the centers of the maximal spheres in X is a region of dimension n. Put $Z = X$ and let Z' be as in the hypothesis. Let f be the isometry such that $Z' = f(Z)$. We show that $Z' = Y$. For this, it suffices to show that $Z' \neq Y$ leads to a contradiction. Choose n maximal spheres of X such that the convex hull of their centers is a region of dimension n. Call W the sum of these spheres. Let $W' = f(W)$. Since $[\![\Sigma\mathsf{SS}]\!](X)$ and the hypothesis on the maximal spheres of X, one must have $Z' \subseteq Y$. Let $Y - Z' \neq \emptyset$. Since the regions Y and Z' are regular and open, one can find a ball $U' \subseteq Y - Z'$ such that no maximal sphere in W' has the diameter of U' and all maximal spheres of W' are maximal in $W' \cup U'$ as well. Furthermore, we take U' such that $Z' \cap U' = \emptyset$ in structures Φ_α and Φ_β. In structures Φ_γ and Φ_δ, we also add a new region $C' \subseteq Y$ that connects W' and U' (this condition is necessary to guarantee that region Z'_U, which we are going to construct, exists in these structures).

Let $Z'_U = Z' \cup U'$ (we use $U' \cup C'$ instead of U' in structures Φ_γ and Φ_δ) and let $U = f^{-1}(U')$. Fix $Z_U = Z \cup U$, i.e., $Z'_U = f(Z_U)$. Since $Z' \subset Z'_U$ and f is an isometry, $Z \subset Z_U$. Thus, $X \subset Z_U$. Now, apply the hypothesis to Z'_U to get a region Z''_U. Since $Z'_U \subseteq Y$, one must have $Z''_U \subseteq X$. By construction and the choices of Z and U, Z' is congruent to X and Z'_U is congruent to Z''_U, thus Z'_U is congruent to Z' (or a part of Z'). But $Z' \subset Z'_U$, a contradiction. We conclude $Z' = Y$, that is, $\mathrm{Congr}(X, Y)$.

Case (III): $[\![\Sigma\mathsf{SS}]\!](X)$ and the convex hull of the centers of the maximal spheres in X is a region of dimension less than n.

We proceed as before, but this time region W must contain some sphere that is disjoint from X. (Again, in structures Φ_γ and Φ_δ, we also consider a region C that connects W' and Y in such a way that the maximal spheres of Y and W' do not change. One gets the conclusion as in Case (II) by considering the isometry f identified by $Z' = f(Z)$.

Case (IV): Not $[\![\Sigma\mathsf{SS}]\!](X)$.

Since X is not the whole space, there exists a sequence Z_i, with $[\![\Sigma\mathsf{SS}]\!](Z_i)$, $X \subset Z_i$, and $\cup Z_{i+1} \subset \cup Z_i$, that converges to X. Let Z'_i be the region satisfying the hypothesis when applied to Z_i and let f_i be the function for which $Z'_i = f_i(Z_i)$. Since $[\![\Sigma\mathsf{SS}]\!](Z_i)$, f_i must be an isometry. Furthermore, $f_{i+1}(\cup Z_{i+1}) \subset f_i(\cup Z_i)$, $f_i(X) \subset f_i(\cup Z_i)$, and, for $i \to +\infty$, $\mathrm{Vol}(\cup Z_i) \to \mathrm{Vol}(X)$. From these, the sequence Z'_i converges to a region (call it Z') containing Y. First, we show that Congr(X, Z'). Assume not, then there exists a set N of points in X (or in Z') such that $f(N) \not\subset Z'$ for all isometries f (analogously $f(N) \not\subset X$ for all isometries f, if N in Z'). Note that $N \subset X \subset Z_i$ for all i. From $f(N) \not\subset Z'$ and the fact that Z' is the limit of $f_i(Z_i)$, there is an index m such that $f_m(N) \not\subset Z'_m$. Let m be the first index for which this happens. By construction, f_m is an isometry and $f_m(X) = Z'_m$, contradicting $f_m(N) \not\subset Z'_m$. We have seen that $Y \subseteq Z'$ and Congr(X, Z'). It remains to show that $Z' - Y \neq \emptyset$ leads to a contradiction. This follows from considering a new sequence W_i, with $[\![\Sigma\mathsf{SS}]\!](W_i)$, $W_i \subset X$, and $W_i \subset W_{i+1}$, that converges to X and the sequence defined by $f(W_i) = W'_i$ for all i, since $f(W_i) \subset Y$ for all i. □

10.2. Proof of Lemma 2. We prove that
(A.1) $[\![\mathsf{C}^*_2(x, y)]\!]_{\alpha-\delta}$ = (dist(X, Y) = 0);
(A.2) $[\![\mathsf{SC}^*_2(x)]\!]_{\alpha-\delta}$ = WWConx(X);
(A.3) $[\![\mathsf{LEDiam}_2(x, y)]\!]_{\alpha-\delta}$ = WWConx$(Y) \wedge$ diam$(X) \leq$ diam(Y);
(A.4) $[\![\mathsf{SCDiam}_2(x, y)]\!]_{\alpha-\delta}$ = WWConx$(X) \wedge$ diam$(X) \leq$ diam(Y);
(A.5) $[\![\mathsf{LDist}_2(x, y, x', y')]\!]_{\alpha-\delta}$ = dist$(X, Y) <$ dist(X', Y');
(A) $[\![\mathsf{CCon}_2(c, x, y)]\!]_{\alpha-\delta}$ = dist$(X, Y) \leq$ diam(C).

Proof. **(A.1)** We have to prove the following equivalence:
dist$(X, Y) = 0 \Leftrightarrow \forall Z \exists Z'($Congr$(Z', Z) \wedge Z' \cap X \neq \emptyset \wedge Z' \cap Y \neq \emptyset)$.
($\Rightarrow$) If dist(X, Y) = 0, then for each $d > 0$ there exist $x \in X$ and $y \in Y$ such that dist$(x, y) < d$. In particular, take a ball $B \subseteq Z$ and consider x and y such that dist$(x, y) <$ diam(B). From L.13, there exist $z, z' \in B$ such that dist(z, z') = dist(x, y). It suffices to consider an isometry f such that $Z' = f(Z)$, $x = f(z)$, and $y = f(z')$.
($\Leftarrow$) (By contradiction) Assume dist$(X, Y) = d > 0$ and consider Z such that diam$(Z) < d$. Clearly, the hypothesis fails. □

(A.2) Directly from (A.1), Proposition 1, and the definition of WWConx.
(A.3) For every w-weakly connected region Y, we have to prove the following equivalence in all the domains:
diam$(X) \leq$ diam$(Y) \Leftrightarrow \forall A,B((A \subseteq X \wedge B \subseteq X) \to \exists Y'($Congr$(Y', Y) \wedge Y' \cap A \neq \emptyset \wedge Y' \cap B \neq \emptyset))$.
($\Rightarrow$) Let $a \in A$ and $b \in B$ be 2 points in X. Since diam$(X) \leq$ diam(Y) and Y is w-weak connected, from L.18 there exist $y,y' \in Y$ such that dist(y, y') = dist(a, b). It is enough to consider an isometry f with $a = f(y)$ and $b = f(y')$ and to take $Y' = f(Y)$.
($\Leftarrow$) (By contradiction) Let us assume diam$(X) >$ diam(Y). It follows that there exist $a,b \in X$ with dist$(a, b) -$ diam$(Y) = \varepsilon > 0$. Let A and B be 2 balls in X with diameter smaller than $\varepsilon/2$ (such A and B exist because X is an open region). Clearly, we have $A \subseteq X$, $B \subseteq X$, and dist$(A, B) >$ diam(Y), a contradiction.
(A.4) For every w-weakly connect region X, we have to prove the following equivalence in all the domains:
diam$(X) \leq$ diam$(Y) \Leftrightarrow \forall Z((Y \subseteq Z \wedge$ WWConx$(Z)) \to$ diam$(X) \leq$ diam$(Z))$.
($\Rightarrow$) Since $Y \subseteq Z$, we have diam$(Y) \leq$ diam(Z), thus (from the hypothesis) diam$(X) \leq$ diam(Z).
($\Leftarrow$) Let Z be the *convex hull* of Y. By the definition of the convex hull, $Y \subseteq Z$, and by L.19, diam(Z) = diam(Y). By L.16, we have Conx(Z) and from L.1, WWConx(Z). It follows that diam$(X) \leq$ diam(Y).

(A.5) We have to prove the following equivalence in all the domains:

$\text{dist}(X, Y) < \text{dist}(X', Y') \Leftrightarrow \exists A(\text{WWConx}(A) \wedge A \cap X \neq \emptyset \wedge A \cap Y \neq \emptyset \wedge \forall A'(\text{Congr}(A', A) \rightarrow (A' \cap X' = \emptyset \vee A' \cap Y' = \emptyset)))$.

($\Rightarrow$) It is enough to consider a ball A overlapping both X and Y with diameter equal to $\text{dist}(X, Y) + \varepsilon$, where $\varepsilon < \text{dist}(X', Y') - \text{dist}(X, Y)$.

($\Leftarrow$) (By contradiction) Let us assume $\text{dist}(X, Y) \geq \text{dist}(X', Y')$. Let A be an arbitrary open region such that WWConx(A) and $x \in \{A \cap X\}$, $y \in \{A \cap Y\}$. Since $\text{dist}(x, y) \geq \text{dist}(X', Y')$, there are $x' \in X'$, $y' \in Y'$ such that $\text{dist}(x', y') \leq \text{dist}(x, y)$. From L.18, there exist $a, a' \in A$ such that $\text{dist}(a, a') = \text{dist}(x', y')$. We obtain a contradiction taking $A' = f(A)$ with f an isometry such that $a = f(x')$ and $a' = f(y')$.

(A) We have to prove the following equivalence in all the domains:
$\text{dist}(X, Y) \leq \text{diam}(C) \Leftrightarrow \forall A, B(\text{dist}(A, B) < \text{dist}(X, Y) \rightarrow \exists Z(\text{WConx}(Z) \wedge \text{diam}(Z) \leq \text{diam}(C) \wedge Z \cap A \neq \emptyset \wedge Z \cap B \neq \emptyset))$.

($\Rightarrow$) For C infinite, take $Z = \mathrm{R}^n$. Let now C be finite, i.e., $\text{diam}(C) < +\infty$. Let A and B be such that $\text{dist}(A, B) < \text{dist}(X, Y)$. From the proof of (A.5), we have $\exists Z(\text{WWConx}(Z) \wedge Z \cap A \neq \emptyset \wedge Z \cap B \neq \emptyset \wedge \forall Z'(\text{Congr}(Z', Z) \rightarrow (Z' \cap X = \emptyset \vee Z' \cap Y = \emptyset)))$. Since $\text{Congr}(Z', Z) \rightarrow \text{diam}(Z') = \text{diam}(Z)$, from $\forall Z'(\text{Congr}(Z', Z) \rightarrow (Z' \cap X = \emptyset \vee Z' \cap Y = \emptyset))$, we have $\text{diam}(Z) \leq \text{dist}(X, Y)$, then, from the hypothesis, $\text{diam}(Z) \leq \text{diam}(C)$. From L.1 and $\text{diam}(C) < +\infty$, WConx(Z).

($\Leftarrow$) (By contradiction) Let us assume $\text{dist}(X, Y) > \text{diam}(C)$. Let $A \subseteq X$ and $B \subseteq Y$ be 2 open regions with $\text{dist}(A, B) - \text{diam}(C) = \varepsilon > 0$. From the definition of dist and diam, if Z is such that $\text{diam}(Z) \leq \text{diam}(C)$, then Z cannot overlap both A and B, a contradiction.

10.3. Proof of Lemma 4. Using (DP) in Proposition 1 and (DFD5) in Dispensable Primitives section, we need to prove that

(A) $[\![\mathsf{Conj}_5(x, y, x', y')]\!]_{\alpha-\delta} = [\![\mathsf{Conj}(x, y, x', y')]\!]$.

We start proving that:

(A.1) $[\![\mathsf{Eq}(z, x, y)]\!]_{\alpha-\delta} = (\text{dist}(Z, X) = \text{dist}(Z, Y))$;
(A.2) $[\![\mathsf{EqD}^*(x, y, x', y')]\!]_{\alpha-\delta} = (\text{dist}(X, Y) = \text{dist}(X', Y')) \wedge \text{diam}(X) < +\infty \wedge \text{diam}(Y) < +\infty \wedge \text{diam}(X') < +\infty \wedge \text{diam}(Y') < +\infty$.

Proof. **(A.1)** It holds because of the following (obvious) equivalence:

$$\text{dist}(Z, X) = \text{dist}(Z, Y) \Longleftrightarrow \neg\text{dist}(Z, X) < \text{dist}(Z, Y) \wedge \neg\text{dist}(Z, Y) < \text{dist}(Z, X). \tag{A.1}$$

□

Regarding **(A.2)**, the 2 alternative definitions (a) and (b) of EqD^* need to be considered.

(A.2.a) Following the definition (a) of EqD^*, assuming X, Y, X', Y' of finite diameter, we need to prove that in Φ_α and Φ_β for R^1 and in $\Phi_{\alpha-\delta}$ for $\mathrm{R}^{n>1}$, the following equivalence holds:

$\text{dist}(X, Y) = \text{dist}(X', Y') \Longleftrightarrow \exists Z, Z'(\text{dist}(X, \text{Y}) = \text{dist}(X, Z) \wedge \text{dist}(X', Y') = \text{dist}(X', Z') \wedge \text{dist}(Z, X) = \text{dist}(Z, Z') \wedge \text{dist}(Z', X') = \text{dist}(Z', Z))$.

($\Rightarrow$) (Sketch) Put $\text{dist}(X, Y) = \text{dist}(X', Y') = d$.

(a) Assume $d = 0$. The thesis follows considering a ball $Z = Z'$ containing $X \cup X'$.

(b) Assume $d > 0$ and $n = 1$. Suppose that there exists $p \in \partial(X)$ such that all the points in $X \cup X'$ lie on the same side of R^1 with respect to p (a similar argument holds for $p' \in \partial(X')$ such that all the points in $X \cup X'$ lie on the same side of R^1 with respect to p).

Let Z be finite and connected such that $\text{dist}(Z, X) = d$ with Z on the opposite side of X with respect to p. Let Z'_1 be finite and connected such that $\text{dist}(Z'_1, Z) = d$ with Z'_1 on the opposite side of p with respect to Z. Let Z'_2 be finite connected such that $\text{dist}(Z'_2, X') = d$ with Z'_2 on the opposite side of p with respect to X'. By construction, $\text{dist}(Z'_1, Z'_2) \geq d$. Then, it suffices to put $Z' = Z'_1 \cup Z'_2$ (since Z' is not connected, this proof holds only in Φ_α and Φ_β).

(c) Assume $d > 0$ and $n > 1$. Consider $X\hat{} = \{x \in \mathrm{R}^n \mid \text{dist}(x, X) < d\}$, $X'\hat{} = \{x \in \mathrm{R}^n \mid \text{dist}(x, X') < d\}$. $X\hat{}$ and $X'\hat{}$ have finite diameter; therefore, $\mathrm{R}^n - [X'\hat{} \cup X'\hat{}]$ has at most one connected component with infinite diameter. Call it V. Let $p \in \partial(V) \cap \partial(X'\hat{})$ (otherwise take $p \in \partial(V) \cap \partial(X\hat{})$ and switch $X'\hat{}$, $X\hat{}$ in the rest of the proof). By construction, there exists $q \in \partial(X\hat{})$ such that q is path-connected to p in $\mathrm{R}^n - X\hat{}$ and $\text{dist}(p, q) > d$. Let Z' be a connected region in V such that $p \in \partial(Z')$ and $\text{diam}(Z) < (d - \text{dist}(p, q))/2$. Let Z be a connected region in $\mathrm{R}^n - X\hat{}$ such that $q \in \partial(Z)$ and $\text{dist}(Z, Z') = d$ (since q is path-connected to p in $\mathrm{R}^n - X\hat{}$ and $\text{dist}(p, q) > d$, this region exists always). Clearly, we have $\text{dist}(Z, X) = \text{dist}(Z, Z') = \text{dist}(Z', X') = d$.

($\Leftarrow$) Trivial.

(A.2.b) In $\Phi_{\gamma,\delta}$ for R^1, we use the definition (b) of EqD^*. First, we prove that

(B.1) $[\![\mathsf{CG}^{\mathsf{S}}(x, y)]\!]_{\gamma,\delta} = (\text{diam}(X) = \text{diam}(Y) < +\infty \wedge [X] \cap [Y] = \varnothing)$;

(B.2) $[\![\mathsf{CG}^*(\mathrm{x,y})]\!]_{\gamma,\delta} = (\text{diam}(X) = \text{diam}(Y) < +\infty)$.

(B.1) Given 2 connected regions X, Y with finite diameter (in $\Phi_{\gamma,\delta}$, all the regions are Conx) such that $[X] \cap [Y] = \varnothing$, we need to prove the following equivalence:
$\text{diam}(X) = \text{diam}(Y) \Leftrightarrow \exists Z_1,Z_2,Z_3([Z_2] \cap [X] \neq \varnothing \wedge Z_2 \cap X = \varnothing \wedge [Z_2] \cap [Y] \neq \varnothing \wedge Z_2 \cap Y = \varnothing \wedge [Z_1] \cap [X] \neq \varnothing \wedge Z_1 \cap X = \varnothing \wedge [Z_1] \cap [Z_2] = \varnothing \wedge [Z_3] \cap [Y] \neq \varnothing \wedge Z_3 \cap Y = \varnothing \wedge [Z_3] \cap [Z_2] = \varnothing \wedge \text{dist}(Z_2, Z_1) = \text{dist}(Z_2, Z_3))$

From the hypothesis, X and Y are connected regions and are not connected to each other. Let $X = (x_1, x_2)$, $Y = (y_1, y_2)$, and assume that x_2 strictly precedes y_1 (a similar argument holds if we take y_2 precedes x_1). Since Z_2 needs to be externally connected to both X and Y and $[X] \cap [Y] = \varnothing$, then $Z_2 = (x_2, y_1)$. $Z_1(Z_3)$ needs to be externally connected to $X(Y)$ and it does not overlap Z_2; therefore, $Z_1 = (x_1 - d_1, x_1)$ ($Z_3 = (y_2, y_2 + d_2)$) for some $d_1 < +\infty$ ($d_2 < +\infty$). By construction, $\text{dist}(Z_2, Z_1) = \text{diam}(X)$ and $\text{dist}(Z_2, Z_3) = \text{diam}(Y)$. **(B.1)**

(B.2) Given 2 connected regions X, Y (in $\Phi_{\gamma,\delta}$, all the regions are Conx) with finite diameter, we need to prove the following equivalence:
$\text{diam}(X) = \text{diam}(Y) \Longleftrightarrow$

(a) $X = Y \vee$

(b) $(X \cap Y \neq \varnothing \wedge \neg X \subseteq Y \wedge \neg Y \subseteq X \wedge \exists Z_1, Z_2(Z_1 = X - [Y] \wedge Z_2 = Y - [X] \wedge [Z_1] \cap [Z_2] = \varnothing \wedge \text{diam}(Z_1) = \text{diam}(Z_2))) \vee$

(c) $([X] \cap [Y] \neq \varnothing \wedge X \cap Y = \varnothing \wedge \neg \exists Z((Z \subset X \wedge \text{diam}(Z) = \text{diam}(Y) \wedge [Z] \cap [Y] = \varnothing) \vee (Z \subset Y \wedge \text{diam}(Z) = \text{diam}(X) \wedge [Z] \cap [X] = \varnothing))) \vee$

(d) $([X] \cap [Y] = \varnothing \wedge \text{diam}(X) = \text{diam}(Y))))$.

On the basis of the *parthood* and *connection* relations, there are a total of 8 distinct cases to consider between 2 connected regions: (1) $\partial(X) \cap \partial(Y) \neq \varnothing$ and $X \subset Y$, (2) $\partial(X) \cap \partial(Y) \neq \varnothing$ and $Y \subset X$, (3) $\partial(X) \cap \partial(Y) = \varnothing$ and $X \subset Y$, (4) $\partial(X) \cap \partial(Y) = \varnothing$ and

$Y \subset X$, (5) $X = Y$, (6) X and Y partially overlap, (7) X and Y are externally connected, and (8) X and Y are not connected. In cases (1)–(4), the regions have necessarily different diameters since we are in R^1. In addition, conditions (a)–(d) correspond to cases (5)–(8); thus, we can exclude cases (1)–(4) altogether.

Let $X = (x_1, x_2)$, $Y = (y_1, y_2)$.

Case (5): Both diam(X) = diam(Y) and condition (a) follow.

Case (6): We have $X \cap Y \neq \emptyset \neg X \subseteq Y \neg \wedge \neg Y \subseteq X$, and since X and Y are connected and finite, then also $Z_1 = X - [Y]$ and $Z_2 = Y - [X]$ are connected and finite. In addition, by definition, we have $[Z_1] \cap [Z_2] = \emptyset$. In this case, diam($X$) = diam($Z_1$) + diam($X \cap Y$) and diam($Y$) = diam($Z_2$) + diam($X \cap Y$), i.e., diam($X$) = diam($Y$) iff diam($Z_1$) = diam($Z_2$).

Case (7): We have $[X] \cap [Y] \neq \emptyset \wedge X \cap Y = \emptyset$; therefore, there exists x such that $[X] \cap [Y] = \{x\}$. If diam(X) = diam(Y), condition (c) is easily verified. For the other direction, condition (c) holds diam(Z) $<$diam(Y) for all $Z \subset X$. Thus, diam(X) $\leq$ diam(Y). Analogously, one shows diam(Y) $\leq$ diam(X).

Case (8): We have $[X] \cap [Y] = \emptyset$. Then, diam(X) = diam(Y) and condition (d) are equivalent. **(B.2)**

Following the definition (b) of EqD^*, assuming X, Y, X', Y' of finite diameter and connected, we have now to prove that in $\Phi_{\gamma,\delta}$ for R^1, the following equivalence holds:

dist(X, Y) = dist(X', Y') $\Leftrightarrow$ ($[X] \cap [Y] \neq \emptyset \wedge [X'] \cap [Y'] \neq \emptyset$) $\vee\ \exists Z,Z'([Z] \cap [X] \neq \emptyset \wedge Z \cap X = \emptyset \wedge [Z] \cap [Y] \neq \emptyset \wedge Z \cap Y = \emptyset \wedge [Z'] \cap [X'] \neq \emptyset \wedge Z' \cap X' = \emptyset \wedge [Z'] \cap [Y'] \neq \emptyset \wedge Z' \cap Y' = \emptyset \wedge$ diam(Z) = diam(Z')).

($\Rightarrow$) Let dist(X, Y) = dist(X', Y'). If $[X] \cap [Y] \neq \emptyset$ and $[X'] \cap [Y'] \neq \emptyset$, we are done. Assume $[X] \cap [Y] = \emptyset$. From L.15, dist(X, Y) $>$ 0 and so dist(X', Y') $>$ 0, i.e., $[X'] \cap [Y'] = \emptyset$. Let $Z = \{z \in \mathrm{R}^1 | z \notin [X \cup Y]$ and Btw(z, x, y) for some $x \in X$, $y \in Y\}$ and $Z' = \{z' \in \mathrm{R}^1 | z' \notin [X' \cup Y']$ and Btw(z', x', y') for some $x' \in X'$, $y' \in Y'\}$. Thus, diam(Z) = dist(X, Y) = dist(X', Y') = diam(Z'), and the other conditions are satisfied by construction.

($\Leftarrow$) (By contradiction) Let dist(X, Y) $>$ dist(X', Y') so that $[X] \cap [Y] = \emptyset$. If there exist Z and Z' externally tangent to X, Y and to X', Y' (respectively) such that diam(Z) = diam(Z'), then, since Z is connected, diam(Z) = dist(X, Y) and diam(Z') = dist(X', Y'). But dist(X, Y) $>$ dist(X', Y') and diam(Z) = diam(Z'), a contradiction. **(A.2.b)/(A.2)**

We now prove that:

(A) $[\![\mathsf{Conj}_5(x, y, x', y')]\!]_{\alpha-\delta} = [\![\mathsf{Conj}(x, y, x', y')]\!]$.

We begin with a lemma:

Lemma C.2.1. $\forall X, Y, Z, W(([X] \cap [Y] \neq \emptyset\ \wedge [Z] \cap [W] \neq \emptyset\ \wedge \mathit{diam}(X) < +\infty\ \wedge \mathit{diam}(Z) < +\infty) \rightarrow \exists x, z(x \in [X] \cap [Y] \wedge\ z \in [Z] \cap [W] \wedge \mathit{dist}(x, z) = \mathit{dist}([X] \cap [Y], [Z] \cap [W])))$.

Proof. diam(X) $< +\infty$ implies diam($[X] \cap [Y]$) $< +\infty$. Furthermore, $[\![X] \cap [Y]\!] = [X] \cap [Y]$. Similarly for $[Z] \cap [W]$. The thesis follows from L.12. ***(Lemma)*** □

(A) It remains to prove the following equivalence that we state somewhat informally in the attempt to improve readability:

$\exists x,y,x',y'(x \in [X] \wedge y \in [Y] \wedge x' \in [X'] \wedge y' \in [Y'] \wedge$ dist(x, y) = dist(x', y')) $\Longleftrightarrow$

There exist 4 finite and connected regions A, B, A', B' (the finiteness of the diameter follows from the EqD^* condition) with properties:

$[A] \cap [X] \neq \emptyset$, $[B] \cap [Y] \neq \emptyset$, $[A'] \cap [X'] \neq \emptyset$, $[B'] \cap [Y'] \neq \emptyset$, dist($A$, B) = dist ($A'$, B');

and such that:

1. $\forall P_A, P_B((P_A \subseteq A \wedge P_B \subseteq B \wedge \text{dist}(P_A, P_B) = \text{dist}(A, B)) \rightarrow [P_A] \cap [X] \neq \emptyset \wedge [P_B] \cap [Y] \neq \emptyset)$.
2. $\forall P_{A'}, P_{B'}((P_{A'} \subseteq A' \wedge P_{B'} \subseteq B' \wedge \text{dist}(P_{A'}, P_{B'}) = \text{dist}(A', B')) \rightarrow [P_{A'}] \cap [X'] \neq \emptyset \wedge [P_{B'}] \cap [Y'] \neq \emptyset)$.

($\Rightarrow$) *Case* $n = 1$: Without loss of generality, assume $x \leq y$ and fix a value $r < +\infty$. Consider the open balls $A = (x - r, x)$ and $B = (y, y + r)$. Clearly, $x \in [A] \cap [X]$, $y \in [B] \cap [Y]$, and condition 1 is satisfied since x must belong to $[P_A]$ and y to $[P_B]$. We can build A' and B' analogously from x', y'. Then, $\text{dist}(A, B) = \text{dist}(A', B')$, $[A'] \cap [X'] \neq \emptyset$, $[B'] \cap [Y'] \neq \emptyset$, and condition 2 is satisfied as well.

Case $n > 1$: Let r be finite.

(a) Assume $x = y$ (and so $x' = y'$), then it suffices to consider $A = \text{ball}(x, r) \cap X$, $B = \text{ball}(x, r) \cap Y$, $A' = \text{ball}(x', r) \cap X'$, and $B' = \text{ball}(x', r) \cap Y'$ (or appropriate connected subregions of these).

(b) Assume $x \neq y$ (and so $x' \neq y'$).

Fix the line l through x and y and 2 finite balls A and B with centers in l and such that $x \in \partial(A)$, $y \in x \in \partial(B)$, and $\text{dist}(A, B)= \text{dist}(x, y)$. Analogously, find A' and B' with centers in l' (the line through x' and y'), $x' \in [A']$, $y' \in [B']$, and $\text{dist}(A', B') = \text{dist}(x', y')$. By construction, we have $[A] \cap [X] \neq \emptyset$, $[B] \cap [Y] \neq \emptyset$, $[A'] \cap [X'] \neq \emptyset$, $[B'] \cap [Y'] \neq \emptyset$, and $\text{dist}(A, \text{B})=\text{dist}(A', B')$. Regarding conditions 1 and 2, observe that only the 2 pairs x, y and x', y', (in A, B and in A', B', respectively) have distance equal to $\text{dist}(A, B)$. Thus, if $P_A \subseteq A \wedge P_B \subseteq B \wedge \text{dist}(P_A, P_B) = \text{dist}(A, B)$ and $P_{A'} \subseteq A' \wedge P_{B'} \subseteq B' \wedge \text{dist}(P_{A'}, P_{B'}) = \text{dist}(A', B')$, we must have $x \in [P_A]$, $y \in [P_B]$, $x' \in [P_{A'}]$, and $y' \in [P_{B'}]$. This guarantees that conditions 1 and 2 are satisfied.

($\Leftarrow$) From the hypothesis, A, B, A', and B' have finite diameter. Also, $[A] \cap [X] \neq \emptyset$ and $[B] \cap [Y] \neq \emptyset$. We show that $\text{dist}([A] \cap [X], [B] \cap [Y]) = \text{dist}(A, B)$. From the definition of dist, $\text{dist}([A] \cap [X], [B] \cap [Y]) \geq \text{dist}(A, B)$. Suppose $\text{dist}([A] \cap [X], [B] \cap [Y]) > \text{dist}(A, B)$. From L.12, there exist $a \in [A]$ and $b \in [B]$ such that $\text{dist}(a, b) = \text{dist}(A, B)$. Let $d = \text{dist}([A] \cap [X], [B] \cap [Y]) - \text{dist}(A, B)$, $S_A = A \cap \text{ball}(a, d/3)$, and $S_B = B \cap \text{ball}(b, d/3)$ (or appropriate connected subregions). Since $\text{diam}(S_A \cup S_B) \leq \text{dist}(A, B) + 2/3d < \text{dist}([A] \cap [X], [B] \cap [Y])$, we have $[S_A] \cap [X] = \emptyset$ or $[S_B] \cap [Y] = \emptyset$. This contradicts condition 1; thus, $\text{dist}([A] \cap [X], [B] \cap [Y]) = \text{dist}(A, B)$. Analogously, we have $\text{dist}([A'] \cap [X'], [B'] \cap [Y']) = \text{dist}(A', B')$. From $\text{dist}(A, B) = \text{dist}(A', B')$, one obtains $\text{dist}([A] \cap [X], [B] \cap [Y]) = \text{dist}([A'] \cap [X'], [B'] \cap [Y'])$. From Lemma C.2.1, there exist x, y, x', and y' such that $x \in [A] \cap [X]$, $y \in [B] \cap [Y]$, $x' \in [A'] \cap [X']$, $y' \in [B'] \cap [Y']$ and $\text{dist}(x, y) = \text{dist}(x', y') = \text{dist}(A, B)$.

10.4. Proof of Lemma 5. We start proving that

(A.1) $[\![\mathsf{FD}_3(x)]\!]_{\alpha-\delta} = \text{diam}(X) < +\infty$.

(A) $[\![\mathsf{C}_3(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{C}(x, y)]\!]$.

Proof. **(A.1)** First, we need to find the interpretation of SC_3^*. From the proof of Proposition 10 (see Appendix B.8), it follows that $[\![\mathsf{C}_3^*(x, y)]\!]_{\alpha-\delta} = \text{dist}(X, Y) = 0$. From Proposition 1 and the definition of WWConx, we directly obtain that $[\![\mathsf{SC}_3^*(x, y)]\!]_{\alpha-\delta} = \text{WWConx}(X)$. Thus, we need to prove the following formula in all the domains:

$\text{diam}(X) < +\infty \Leftrightarrow$

$\exists X', Y, Z(\text{WWconx}(X') \wedge X \subseteq X' \wedge \neg\exists x, x', y, z(x, x' \in [X'] \wedge y \in [Y] \wedge z \in [Z] \wedge \text{dist}(x, x') = \text{dist}(y, z)))$.

($\Rightarrow$) If $d = \text{diam}(X) < +\infty$, it is sufficient to consider 3 balls X', Y, and Z of finite diameter $d' \geq d$ such that $X \subseteq X'$ and $\text{dist}(Y, Z) > d'$.

($\Leftarrow$) (By contradiction) Let $\mathrm{diam}(X) = +\infty$ and $\mathrm{WWconx}(X') \wedge X \subseteq X'$. Clearly, $\mathrm{diam}(X') = +\infty$. From L.18, for all $d \in \mathrm{R}$, there exist $x,x' \in [X']$ such that $\mathrm{dist}(x, x') = d$. Fix any Y and Z. Then, for all $y \in [Y]$ and $z \in [Z]$, we can find $x,x' \in [X']$ such that $\mathrm{dist}(x, x') = \mathrm{dist}(y, z)$, a contradiction. □

Proof. **(A)** Let us observe that $[\![\mathsf{FD}_3(x)]\!]_{\alpha-\delta} = [\![\mathsf{FD}_5(x)]\!]_{\alpha-\delta}$, $[\![\mathsf{P}_5(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{P}(x, y)]\!]$ $[\![\mathsf{C}^*_3(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{C}^*_5(x, y)]\!]_{\alpha-\delta}$ and that the definition of C_3 is analogous to the one of C_5. It is possible to consider the same proof given for C_5 (see (DC5) in Dispensable Primitives section) in Appendix B.9. □

Regarding S_3, we use the results above and prove that

(B.1) $[\![\mathsf{LEDiam}_3(x, y)]\!]_{\alpha-\delta} = \mathrm{Conx}(X) \wedge \mathrm{Conx}(Y) \wedge \mathrm{diam}(X) \leq \mathrm{diam}(Y)$;

(B) $[\![\mathsf{S}_3(x)]\!]_{\alpha-\delta} = [\![\mathsf{S}(x)]\!] = \exists c,r(X = \mathrm{Ball}(c, r))$.

Proof. **(B.1)** Given 2 connected open regions X and Y, we have to prove the following equivalence in all the domains:

$\mathrm{diam}(X) \leq \mathrm{diam}(Y) \Leftrightarrow \forall A,B((A \subseteq X \wedge B \subseteq X) \rightarrow \exists A',B'(A' \subseteq Y \wedge B' \subseteq Y \wedge \exists a,b,a',b'(a \in [A] \wedge b \in [B] \wedge a' \in [A'] \wedge b' \in [B'] \wedge \mathrm{dist}(a, b) = \mathrm{dist}(a', b'))))$.

($\Rightarrow$) Since A, B are subsets of X and A', B' are subsets of Y, it is enough to show that $\forall a,b(a,b \in [X] \rightarrow \exists a',b'(a',b' \in [Y] \wedge \mathrm{dist}(a, b) = \mathrm{dist}(a', b')))$. The latter is an immediate consequence of the hypothesis, L.1, and L.13.

($\Leftarrow$) (By contradiction) Assume that there exist 2 connected regions X, Y such that $\mathrm{diam}(X) > \mathrm{diam}(Y)$. By the definition of diam, there exist $a,b \in [X]$ such that for all $a',b' \in [Y]$, $\mathrm{dist}(a, b) > \mathrm{dist}(a', b')$. Let $3d < \mathrm{dist}(a, b) - \mathrm{dist}(a', b')$, we reach a contradiction considering $A = X \cap \mathrm{ball}(a, d)$, $B = X \cap \mathrm{ball}(a', d)$ (or some connected parts of them in the case of domains with only connected regions). □

Proof. (**B**) We have to prove the following equivalence in all the domains:

$\exists c,r(X = \mathrm{Ball}(c, r)) \Longleftrightarrow \mathrm{diam}(X) < +\infty \wedge \mathrm{Conx}(X) \wedge \forall A(\mathrm{Conx}(A) \wedge \mathrm{diam}(A) \leq \mathrm{diam}(X) \rightarrow \exists B(\mathrm{Conx}(B) \wedge \mathrm{diam}(B) \leq \mathrm{diam}(A) \wedge B \subset X \wedge \forall C,D([C] \cap [X] \neq \emptyset \wedge C \cap X = \emptyset \wedge [D] \cap [X] \neq \emptyset \wedge D \cap X = \emptyset) \rightarrow \exists x,y,x',y'(x \in [B] \wedge y \in [C] \wedge x' \in [B] \wedge y' \in [D] \wedge \mathrm{dist}(x, y) = \mathrm{dist}(x', y')))$.

($\Rightarrow$) Given a region A of diameter l, it suffices to take a ball B of diameter l (or less) concentric to X. Since the distance of the boundary of B to a region externally connected to X is always $r - l$, the result follows for every C, D.

($\Leftarrow$) We proceed by contradiction. Let X be connected with finite diameter. Recall that if there exists a point x equidistant to any point in $\partial(X)$, X is a sphere with center x. Assume that X is not a sphere, i.e., for each point x of X, there are at least 2 points $y,z \in \partial(X)$ such that $\mathrm{dist}(x, y) \neq \mathrm{dist}(x, z)$.

Let $F(x) = \max_{y,z \in \partial(X)} (|\mathrm{dist}(x, y) - \mathrm{dist}(x, z)|)$ and fix $a \in X$ such that $F(a)$ is minimum, i.e., for all $x \in X$, $F(a) \leq F(x)$. Of course, $F(a) > 0$ since $[X]$ is compact and is not a sphere. Also, let $c,d \in \partial(X)$ be such that $F(a) = |\mathrm{dist}(a, c) - \mathrm{dist}(a, d)|$. (Again, c,d exist since $\partial(X)$ is compact.) Now, choose a region A with $\mathrm{diam}(A) < F(a)/2$ and fix any $B \subset X$ as in the hypothesis. Fix x in B. By construction, there exist 2 regions C, D with $c \in C$, $d \in D$ that are externally connected to X and such that $|\mathrm{dist}(x,C) - \mathrm{dist}(x,D)| \geq F(a)$. Since $\mathrm{diam}(B) < F(a)/2$, for any point $y \in B$, $\mathrm{dist}(x, C) + F(a)/4 \geq \mathrm{dist}(y, C) \geq \mathrm{dist}(x, C) - F(a)/4$. Then, for any $y,z \in B$, $|\mathrm{dist}(y, C) - \mathrm{dist}(z, D)| \geq F(a) - F(a)/2 > 0$, a contradiction. □(**B**)

10.5. Proof of Lemma 6. As before, we rely on Tarski (1956a) for the definition of the between relation BTW. Note that we write $\mathrm{Btw}(c_1, c_2, c_3)$ for 'c_1 is between c_2 and c_3', which corresponds to $\mathrm{Btw}(c_2, c_1, c_3)$ in Tarski's terminology. Using (DO), (D+), (DPP), and the results in Verifying the Given Explicit Definitions and Dispensable Primitives sections, it remains to prove that

(A.1) $[\![\mathsf{Conv}(x)]\!]_{\alpha-\delta} = Conv(X)$;

(A) $[\![\mathsf{ConvH}_1(x, y)]\!]_{\alpha-\delta} = [\![\mathsf{ConvH}(x, y)]\!] = \mathrm{Conv}(X) \wedge Y \subseteq X \wedge \neg\exists Z(\mathrm{Conv}(Z) \wedge Y \subseteq Z \wedge Z \subset X)$.

Proof. **(A.1)** We prove the following equivalence for $\mathsf{Conv}(x)$:

$\forall x,y,z((x,y \in X \wedge \mathrm{Btw}(z, x, y)) \rightarrow z \in X) \Leftrightarrow \forall S_1,S_2,S_3,c_1,c_2,c_3,r_1,r_2,r_3(S_1 = \mathrm{ball}(c_1, r_1) \wedge S_2 = \mathrm{ball}(c_2, r_2) \wedge S_3 = \mathrm{ball}(c_3, r_3) \wedge S_1 \cup S_2 \subseteq X \wedge \mathrm{Btw}(c_3, c_1, c_2)) \rightarrow S_3 \cap X \neq \emptyset$; the proof of (A) follows trivially.

($\Rightarrow$) Consider $S_1 = \mathrm{ball}(c_1, r_1)$, $S_2 = \mathrm{ball}(c_2, r_2)$, $S_3 = \mathrm{ball}(c_3, r_3)$, $S_1 \cup S_2 \subseteq X$, and $\mathrm{Btw}(c_3, c_1, c_2)$. Since $S_1 \cup S_2 \subseteq X$, then $c_1, c_2 \in X$ and, from the hypothesis, $c_3 \in X$. Then, $S_3 \cap X \neq \emptyset$.

($\Leftarrow$) (By contradiction) Suppose that there exist $x \in X$, $y \in X$, and $z \notin X$ such that $\mathrm{Btw}(z, x, y)$. Thus, $x \neq z \neq y \neq x$. We show that $\exists x',y',z'(\mathrm{Btw}(z', x', y') \wedge x', y' \in X \wedge z' \notin [X])$. From this, the contradiction follows taking the radius of S_3 to be less than $\mathrm{dist}(z', X)$.

The only case to consider is $z \in \partial(X)$. Since X is open, there exist 3 congruent balls $S_x = \mathrm{ball}(x, r)$, $S_y = \mathrm{ball}(y, r)$, and $S_z = \mathrm{ball}(z, r)$ such that $S_x \subseteq X$, $S_y \subseteq X$, $\neg S_z \subseteq [X]$, $S_x \cap S_y = \emptyset$, $S_x \cap S_z = \emptyset$, and $S_y \cap S_z = \emptyset$. Fix a point z' in $S_z - [X]$ and call l the line through x, y, z and l' line through z' and parallel to l. Consider $x' \in S_x \cap l'$ and $y' \in S_y \cap l'$. Since S_x, S_y, and S_z are disjoint congruent balls with aligned centers, from $\mathrm{Btw}(z, x, y)$ we conclude $\mathrm{Btw}(z', x', y')$. Finally, fix 3 balls $S_1 = \mathrm{ball}(x', r_1)$, $S_2 = \mathrm{ball}(y', r_2)$, and $S_3 = \mathrm{ball}(z', r_3)$ such that $S_1 \subseteq X$, $S_2 \subseteq X$, $\neg S_3 \subseteq [X]$, and $S_1 \cap S_2 = \emptyset$. This is possible since X is open and $z \notin [X]$.

□

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Table of definitions.

$\mathsf{C}_1(x, y) =_{\mathrm{def}} \exists z(\mathsf{S}(z) \wedge \forall z'(\mathsf{CNC}(z', z) \rightarrow (\mathsf{O}(z', x) \wedge \mathsf{O}(z', y))))$

$\mathsf{C}_2(x, y) =_{\mathrm{def}} \exists z(\mathsf{S}^*_2(z) \wedge \forall z'(\mathsf{CNC}(z', z) \rightarrow (\mathsf{O}(z', x) \wedge \mathsf{O}(z', y))))$

$\mathsf{C}^*_2(x, y) =_{\mathrm{def}} \forall z \exists z'(\mathsf{CG}(z', z) \wedge \mathsf{O}(z', x) \wedge \mathsf{O}(z', y))$

$\mathsf{C}_3(x, y) =_{\mathrm{def}} \exists z, w(\mathsf{FD}_3(z) \wedge \mathsf{FD}_3(w) \wedge \mathsf{P}(z, x) \wedge \mathsf{P}(w, y) \wedge \mathsf{C}^*_3(z, w))$

$\mathsf{C}^*_3(x, y) =_{\mathrm{def}} \forall z(\mathsf{Conj}(z, z, x, y))$

$\mathsf{C}^*_4(x, y) =_{\mathrm{def}} \forall z(\mathsf{CCon}(z, x, y))$

$\mathsf{C}_5(x, y) =_{\mathrm{def}} \exists z, w(\mathsf{FD}_5(z) \wedge \mathsf{FD}_5(w) \wedge \mathsf{P}_5(z, x) \wedge \mathsf{P}_5(w, y) \wedge \mathsf{C}^*_5(z, w))$

$\mathsf{C}^*_5(x, y) =_{\mathrm{def}} \neg \exists z(\mathsf{Closer}(x, z, y))$

$\mathsf{CCon}_1(z, x, y) =_{\mathrm{def}} \exists z'(\mathsf{CG}(z', z) \wedge \mathsf{C}_1(z', x) \wedge \mathsf{C}_1(z', y))$

$\mathsf{CCon}_2(c, x, y) =_{\mathrm{def}} \forall a, b(\mathsf{LDist}_2(a, b, x, y) \rightarrow \exists z(\mathsf{SCDiam}_2(z, c) \wedge \mathsf{O}(z, a) \wedge \mathsf{O}(z, b)))$

$\mathsf{CG}_1(x, y) =_{\mathrm{def}} \forall z(\Sigma\mathsf{SS}(z) \rightarrow \exists z'(\Sigma\mathsf{CG}(z, z') \wedge \forall s, s'((\mathsf{MSP}(s, z) \wedge \mathsf{MSP}(s', z') \wedge$

$\mathsf{SCG}(s, s')) \to ((\mathsf{P}(s, x) \leftrightarrow \mathsf{P}(s', y)) \wedge (\mathsf{P}(s, y) \leftrightarrow \mathsf{P}(s', x)) \wedge (\mathsf{PO}(s, x) \leftrightarrow \mathsf{PO}(s', y)) \wedge (\mathsf{PO}(s, y) \leftrightarrow \mathsf{PO}(s', x))))))$

$\mathsf{CG}^*(x, y) =_{\mathrm{def}} \mathsf{FD}_5(x) \wedge \mathsf{FD}_5(y) \wedge$
$(x = y \vee$
$(\mathsf{PO}(x, y) \wedge \exists z_1, z_2(\mathsf{DIF}(z_1, x, y) \wedge \mathsf{DIF}(z_2, y, x) \wedge \mathsf{CG}^s(z_1, z_2))) \vee$
$(\mathsf{EC}(x, y) \wedge \neg\exists z((\mathsf{PP}(z, x) \wedge \mathsf{CG}^{\mathsf{s}}(z, y)) \vee (\mathsf{PP}(z, y) \wedge \mathsf{CG}^s(z, x)))) \vee$
$(\neg\, \mathsf{C}(x, y) \wedge \mathsf{CG}^{\mathsf{s}}(x, y)))$

$\mathsf{CG}^{\mathsf{s}}(x, y) =_{\mathrm{def}} \mathsf{FD}_5(x) \wedge \mathsf{FD}_5(y) \wedge \neg\mathsf{C}(x, y) \wedge \exists z_1, z_2, z_3(\mathsf{EC}(z_2, x) \wedge \mathsf{EC}(z_2, y) \wedge \mathsf{EC}(z_1, x) \wedge \neg\mathsf{C}(z_1, z_2) \wedge \mathsf{EC}(z_3, y) \wedge \neg\mathsf{C}(z_3, z_2) \wedge \mathrm{Eq}(z_2, z_1, z_3))$

$\mathsf{Closer}_4(z, x, y) =_{\mathrm{def}} \exists a(\mathsf{CCon}(a, z, x) \wedge \neg\mathsf{CCon}(a, z, y))$

$\mathsf{Compl}_6(y, x) =_{\mathrm{def}} \forall z(\mathsf{C}(z, y) \leftrightarrow \neg\mathsf{IP}(z, x))$

$\mathsf{Conj}_5(x, y, x', y') =_{\mathrm{def}} \exists a, b, a', b'(\mathsf{SR}(a) \wedge \mathsf{SR}(b) \wedge \mathsf{SR}(a') \wedge \mathsf{SR}(b') \wedge \mathsf{C}(a, x) \wedge \mathsf{C}(b, y) \wedge \mathsf{C}(a', x') \wedge \mathsf{C}(b', y') \wedge \mathsf{EqD}^*(a, b, a', b') \wedge \forall p_{\mathrm{a}}, p_{\mathrm{b}}((\mathsf{P}(p_{\mathrm{a}}, a) \wedge \mathsf{P}(p_{\mathrm{b}}, b) \wedge \mathsf{EqD}^*(p_{\mathrm{a}}, p_{\mathrm{b}}, a, b)) \to (\mathsf{C}(p_{\mathrm{a}}, x) \wedge \mathsf{C}(p_{\mathrm{b}}, y)) \wedge \forall p'_{\mathrm{a}}, p'_{\mathrm{b}}((\mathsf{P}(p'_{\mathrm{a}}, a') \wedge \mathsf{C}(p'_{\mathrm{b}}, b') \wedge \mathsf{EqD}^*(p'_{\mathrm{a}}, p'_b, a', b')) \to (\mathsf{C}(p'_{\mathrm{a}}, x') \wedge \mathsf{C}(p'_{\mathrm{b}}, y'))$

$\mathsf{Conv}(x) =_{\mathrm{def}} \forall s_1, s_2, s_3((\mathsf{P}(s_1, x) \wedge \mathsf{P}(s_2, x) \wedge \mathsf{BTW}(s_3, s_1, s_2)) \to \mathsf{O}(s_3, x))$

$\mathsf{ConvH}_1(x, y) =_{\mathrm{def}} \mathsf{Conv}(x) \wedge \mathsf{P}(y, x) \wedge \neg\exists z(\mathsf{Conv}(z) \wedge \mathsf{P}(y, z) \wedge \mathsf{PP}(z, x))$

$\mathsf{DIF}(z, x, y) =_{\mathrm{def}} \forall w(\mathsf{P}(w, z) \leftrightarrow (\mathsf{P}(w, x) \wedge \neg\mathsf{O}(w, y)))$

$\mathsf{EC}(x, y) =_{\mathrm{def}} \mathsf{C}(x, y) \wedge \neg\mathsf{O}(x, y)$

$\mathsf{Eq}(z, x, y) =_{\mathrm{def}} \neg\mathsf{Closer}(z, x, y) \wedge \neg\mathsf{Closer}(z, y, x)$

$\mathsf{EqD}(x, y, x', y') =_{\mathrm{def}} \mathsf{SCG}(x, x') \wedge \mathsf{SCG}(y, y') \wedge \neg\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x) \wedge \neg\mathsf{P}(x', y') \wedge \neg\mathsf{P}(y', x') \wedge \exists z, w(\mathsf{ID}(z, x, y) \wedge \mathsf{ID}(w, x', y') \wedge \mathrm{SCG}(z, w)))$

$\mathsf{EqD}^*(x, y, x', y') =_{\mathrm{def}} \mathsf{FD}_5(x) \wedge \mathsf{FD}_5(y) \wedge \mathsf{FD}_5(x') \wedge \mathsf{FD}_5(y') \wedge \exists z, z'(\mathsf{Eq}(x, y, z) \wedge \mathsf{Eq}(x', y', z') \wedge \mathrm{q}(z, x, z') \wedge \mathsf{Eq}(z', x', z))$ (in $\Phi_{\alpha-\beta}$ for $\mathrm{R}^1$and in $\Phi_{\alpha-\delta}$ for $\mathrm{R}^{n>1}$)

$\mathsf{EqD}^*(x, y, x', y') =_{\mathrm{def}} (\mathsf{C}(x, y) \wedge \mathsf{C}(x', y')) \vee \exists z, z'(\mathsf{EC}(z, x) \wedge \mathsf{EC}(z', y') \wedge \mathsf{CG}^*(z, z')))$ (in $\Phi_{\gamma-\delta}$ for R^1)

$\mathsf{FD}_3(x) =_{\mathrm{def}} \exists x', y, z(\mathsf{SC}^*_3(x') \wedge \mathsf{P}(x, x') \wedge \neg\mathsf{Conj}(x', x', y, z))$

$\mathsf{FD}_5(x) =_{\mathrm{def}} \exists z(\forall x', x''((\mathsf{P}_5(x', x) \wedge \mathsf{P}_5(x'', x)) \to \mathsf{Closer}(x', x'', z)))$

$\mathsf{IP}(x, y) =_{\mathrm{def}} \mathsf{P}(x, y) \wedge \forall z(\mathsf{C}(z, x) \to \mathsf{O}(z, y))$

$\mathsf{LDist}_2(x, y, x', y') =_{\mathrm{def}} \exists a(\mathsf{SC}^*_2(a) \wedge \mathsf{O}(a, x) \wedge \mathsf{O}(a, y) \wedge \forall a'(\mathsf{CG}(a', a) \to (\neg\mathsf{O}(a', x') \vee \neg\mathsf{O}(a', y'))))$

$\mathsf{LEDiam}_2(x, y) =_{\mathrm{def}} \mathsf{SC}^*_2(y) \wedge \forall a, b((\mathsf{P}(a, x) \wedge \mathsf{P}(b, x)) \to \exists y'(\mathsf{CG}(y', y) \wedge \mathsf{O}(y', a) \wedge \mathsf{O}(y', b)))$

$\mathsf{LEDiam}_3(x, y) =_{\mathrm{def}} \mathsf{SR}(x) \wedge \mathsf{SR}(y) \wedge \forall a, b((\mathsf{P}(a, x) \wedge \mathsf{P}(b, x)) \to \exists a', b'(\mathsf{P}(a', y) \wedge \mathsf{P}(b', y) \wedge \mathsf{Conj}(a, b, a', b')))$

$\mathsf{MSP}(x, y) =_{\mathrm{def}} \mathsf{S}(x) \wedge \mathsf{P}(x, y) \wedge \forall z((\mathsf{S}(z) \wedge \mathsf{PP}(x, z)) \to \neg\mathsf{P}(z, y))$

$\mathsf{O}(x, y) =_{\mathrm{def}} \exists z(\mathsf{P}(z, x) \wedge \mathsf{P}(z, y))$

$\mathsf{P}^*(x, y) =_{\mathrm{def}} \forall w(\mathsf{C}(x, w) \to \mathsf{C}(y, w))$

$\mathsf{P}_3(x, y) =_{\mathrm{def}} \forall z(\mathsf{C}^*_3(z, x) \to \mathsf{C}^*_3(z, y))$

$\mathsf{P}^*_4(x, y) =_{\mathrm{def}} \forall z(\mathsf{C}^*_3(z, x) \to \mathsf{C}^*_4(z, y))$

$\mathsf{P}^+_4(x, y) =_{\mathrm{def}} \forall z, w(\mathsf{CCon}(w, z, x) \to \mathsf{CCon}(w, z, y))$

$\mathsf{P}_5(x, y) =_{\mathrm{def}} \forall z(\mathsf{C}^*_5(z, x) \to \mathsf{C}^*_5(z, y))$

$\mathsf{P}_6(x, y) =_{\mathrm{def}} \forall z(\mathsf{C}(z, x) \to \mathsf{C}(z, y))$

$\mathsf{PO}(x, y) =_{\mathrm{def}} \mathsf{O}(x, y) \wedge \neg\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$

$\mathsf{PP}(x, y) =_{\mathrm{def}} \mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$

$\mathsf{S}^*_2(x) =_{\text{def}} \mathsf{SR}(x) \ \wedge \forall y, z((\mathsf{CG}(x, y) \wedge \mathsf{PO}(x, y) \wedge \mathsf{DIF}(z, x, y)) \rightarrow \mathsf{SR}(z))$
$\mathsf{S}_3(x) =_{\text{def}} \mathsf{FD}_3(x) \wedge \mathsf{SR}(x) \ \wedge \forall a(\mathsf{LEDiam}_3(a, x) \rightarrow$
$\quad \exists b(\mathsf{LEDiam}_3(b, a) \wedge \mathsf{P}(b, x) \ \wedge \forall c, d(\mathsf{EC}(c, x) \wedge \mathsf{EC}(d, x)) \rightarrow (\mathsf{Conj}(b, c, b, d)))$
$\mathsf{SC}(x) =_{\text{def}} \forall y, z(\mathsf{SUM}(x, y, z) \rightarrow \mathsf{C}(y, z))$
$\mathsf{SC}^*_2(x) =_{\text{def}} \forall y, z(\mathsf{SUM}(x, y, z) \rightarrow \mathsf{C}^*_2(y, z))$
$\mathsf{SC}^*_3(x) =_{\text{def}} \forall y, z(\mathsf{SUM}(x, y, z) \rightarrow \mathsf{C}^*_3(y, z))$
$\mathsf{SCDiam}_2(x, y) =_{\text{def}} \mathsf{SC}^*_2(x) \ \wedge \forall z((\mathsf{P}(y, z) \wedge \mathsf{SC}^*_2(z)) \rightarrow \mathsf{LEDiam}_2(x, z))$
$\mathsf{SCG}(x, y) =_{\text{def}} \mathsf{S}(x) \wedge \mathsf{S}(y) \ \wedge (x = y \vee \exists z, w(\mathsf{CNC}(z, w) \wedge \mathsf{EC}(z, x) \wedge$
$\quad \mathsf{EC}(z, y) \wedge \mathsf{TPP}(x, w) \wedge \mathsf{TPP}(y, w))$
$\mathsf{SR}_1(x) =_{\text{def}} \forall y, z(\mathsf{SUM}(x, y, z) \rightarrow \exists s(\mathsf{S}(s) \wedge \mathsf{O}(s, y) \wedge \mathsf{O}(s, z) \wedge \mathsf{P}(s, x)))$
$\mathsf{SR}_6(x) =_{\text{def}} \forall y, z, w((\mathsf{SUM}(x, y, z) \wedge \mathsf{Compl}_6(w, x)) \rightarrow$
$\quad \exists v(\mathsf{SC}(v) \wedge \mathsf{O}(v, y) \wedge \mathsf{O}(v, z) \ \wedge \neg\mathsf{C}(v, w))$
$\mathsf{SUM}(z, x, y) =_{\text{def}} \forall w(\mathsf{O}(w, z) \leftrightarrow (\mathsf{O}(w, x) \vee \mathsf{O}(w, y)))$
$\mathsf{TPP}(x, y) =_{\text{def}} \mathsf{PP}(x, y) \wedge \exists z(\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y))$
$\Sigma\mathsf{CG}(x, y) =_{\text{def}} \Sigma\mathsf{SS}(x) \ \wedge \Sigma\ \mathsf{SS}(y) \wedge$
$\quad \forall s(\mathsf{MSP}(s, x) \rightarrow \exists s'(\mathsf{MSP}(s', y) \wedge \mathsf{SCG}(s, s'))) \ \wedge$
$\quad \forall s(\mathsf{MSP}(s, y) \rightarrow \exists s'(\mathsf{MSP}(s', x) \wedge \mathsf{SCG}(s, s'))) \ \wedge$
$\quad \forall s, u, s', u \ (\mathsf{MSP}(s, x) \wedge \mathsf{MSP}(u, x) \wedge \mathsf{MSP}(s', y) \wedge \mathsf{MSP}(u', y)$
$\quad\ \wedge \mathsf{SCG}(s, s') \wedge \mathsf{SCG}(u, u')) \rightarrow \mathsf{EqD}(s, u, s', u')$
$\Sigma\mathsf{SS}(x) =_{\text{def}} \forall y(\mathsf{P}(y, x) \rightarrow \exists s(\mathsf{MSP}(s, x) \wedge \mathsf{O}(s, y)) \ \wedge$
$\quad \forall u, w((\mathsf{MSP}(u, x) \wedge \mathsf{MSP}(w, x) \ \wedge u \neq w) \rightarrow \neg\mathsf{SCG}(u, w))$

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