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1. Generalizations of the G schema

◆ DEFINITION: Where φ any modality (¬ , ◇, or □):

if n=0	$\varphi^n A = A$
if $n = k$+1	$\varphi^n A = \varphi\varphi^k A$

◆ FACT: Consider the schema

$\mathbf{G}^{k,l,m,n} = \Diamond^k \Box^l A \rightarrow \Box^m \Diamond^n A$

Then:

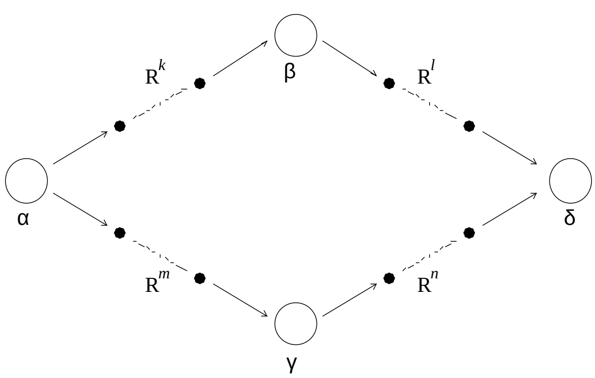
G	=	$\Diamond\Box A \rightarrow \Box\Diamond A$	is just	$\mathbf{G}^{1,1,1,1}$
D	=	$\Box A \rightarrow \Diamond A$	is just	$\mathbf{G}^{0,1,0,1}$
T	=	$\Box A \rightarrow A$	is just	$\mathbf{G}^{0,1,0,0}$
B	=	$A \rightarrow \Box\Diamond A$	is just	$\mathbf{G}^{0,0,1,1}$
4	=	$\Box A \rightarrow \Box\Box A$	is just	$\mathbf{G}^{0,1,2,0}$
5	=	$\Diamond A \rightarrow \Box\Diamond A$	is just	$\mathbf{G}^{1,0,1,1}$

◆ DEFINITION

if n=0	$\alpha R^n \beta \Leftrightarrow \alpha = \beta$
if $n = k$+1	$\alpha R^n \beta \Leftrightarrow \alpha R \gamma$ for some γ W such that $\gamma R^k \beta$

◆ DEFINITION

A standard model $\mathcal{M}$ = W, R, P , is k,l,m,n-incestual iff $\alpha R^k \beta$ & $\alpha R^m \gamma$ $\delta(\beta R^l \delta$ & $\gamma R^n \delta)$



So in particular:

$\mathscr{M}$ is incestual	iff	$\mathscr{M}$ is 1111-incestual
$\mathscr{M}$ is serial	iff	$\mathscr{M}$ is 0101-incestual
$\mathscr{M}$ is reflexive	iff	$\mathscr{M}$ is 0100-incestual
$\mathscr{M}$ is symmetric	iff	$\mathscr{M}$ is 0011-incestual
$\mathscr{M}$ is transitive	iff	$\mathscr{M}$ is 0120-incestual
$\mathscr{M}$ is euclidean	iff	$\mathscr{M}$ is 1011-incestual

(The proofs of these equivalences se are just derivations in first order logic with identity.)

◆ EXAMPLE: $\mathscr{M}$ is serial iff $\mathscr{M}$ is 0101-incestual

Proof: $k \quad m \quad l \quad n$

$\mathscr{M}$ is 0101-incestual	$\alpha \;\; \beta \;\; \gamma[\alpha R^0\beta \;\&\; \alpha R^0\gamma \quad \delta(\beta R^1\delta \;\&\; \gamma R^1\delta)]$	def.
	$\alpha \;\; \beta \;\; \gamma[\alpha=\beta \;\&\; \alpha=\gamma \quad \delta(\beta R\delta \;\&\; \gamma R\delta)]$	def.
	$\alpha=\alpha \;\&\; \alpha=\alpha \quad \delta(\alpha R\delta \;\&\; \alpha R\delta)$	elim
	$\alpha=\alpha \quad \delta(\alpha R\delta)$	& idem
	$\delta(\alpha R\delta)$	since $\alpha=\alpha$
	$\alpha \;\; \delta(\alpha R\delta)$	intro
	$\mathscr{M}$ is serial	def.
$\mathscr{M}$ is serial	$\alpha \;\; \delta(\alpha R\delta)$	def.
	$\delta(\alpha R\delta)$	elim
	$\delta(\alpha R\delta \;\&\; \alpha R\delta)$	& idem
	$\alpha=\beta \;\&\; \alpha=\gamma \quad \delta(\beta R\delta \;\&\; \gamma R\delta)$	= laws
	$\alpha \;\; \beta \;\; \gamma[\alpha=\beta \;\&\; \alpha=\gamma \quad \delta(\beta R\delta \;\&\; \gamma R\delta)]$	intro
	$\alpha \;\; \beta \;\; \gamma[\alpha R^0\beta \;\&\; \alpha R^0\gamma \quad \delta(\beta R^1\delta \;\&\; \gamma R^1\delta)]$	def.
	$\mathscr{M}$ is 0101-incestual	def.

◆ FACT

The schema $\mathbf{G}^{k,l,m,n}$ is valid in the class of all *k*,*l*,*m*,*n*-incestual standard models

◆ COROLLARY

The schema	**G**	is valid in the class of all	**incestual**	standard models
The schema	**D**	is valid in the class of all	**serial**	standard models
The schema	**T**	is valid in the class of all	**reflexive**	standard models
The schema	**B**	is valid in the class of all	**symmetric**	standard models
The schema	**4**	is valid in the class of all	**transitive**	standard models
The schema	**5**	is valid in the class of all	**euclidean**	standard models

◆ $\mathbf{G}^{k,l,m,n}$ is not the most general schema.
For instance, the following are <u>not</u> instances of $\mathbf{G}^{k,l,m,n}$:

$\mathbf{G}_c$	$\Box \Diamond A \rightarrow \Diamond \Box A$
Gr	$\Box (\Box A \rightarrow A) \rightarrow \Box A$

◆ Indeed there are more general schemes with interesting properties—e.g.

Sahl $\Box^{n} (A \rightarrow B)$ (with resrtictions on the form of A and B)

But Gr and Gc are still not covered by such a schema.

2. Characterizability (for Kripkean modal logics)

◆ QUESTION 1:
Does every modal formula correspond to some <u>first-order definable</u> R?
i.e., given a formula A, is there always a first-order sentence φ so that, for every $\mathcal{M} = W, R, P$

$\models^{\mathcal{M}} A$ (modally) iff $\models^{\mathcal{M}} \varphi$ (quantificationally) ?

ANSWER IS NO

— $\mathbf{G}^{k,l,m,n}$	YES	α β γ[αR^{k}β & αR^{m}γ δ(βR^{l}δ & γR^{n}δ)]
— **Sahl**	YES	complicated condition
— **Gr**	NO	there is a condition on R (see test), but <u>not first-order</u> definable
— $\mathbf{G}_c$	NO	not first-order definable (though $\mathbf{G}_c$ **4** is)

◆ QUESTION 2:
What about the other way around? Does every R correspond to a modal formula?

ANSWER IS NO

— E.g.	*Reflexivity*	α(αRα)	$\Box A \rightarrow A$
	Irreflexivity	α(¬α Rα)	no characteristic wff i.e., if a wff is true in every irreflexive model, then it is true in every model

— Ditto for

Asymmetry	α(αRβ → ¬βRα)
Antisymmetry	α(αRβ & βRα → α=β)
Intransitivity	α(αRβ & βRγ → ¬α Rγ)

3. Axiomatic systems (for Kripkean modal logics)

- A system of modal logic is <u>normal</u> iff it contains every instance of

  Df◇ $\Diamond A \leftrightarrow \neg\Box\neg A$
  K $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

  and is closed under the rule

  RN $\dfrac{A}{\Box A}$

- Theorem: Every normal system Σ of modal logic satisfies the *Principle of Duality*:

  $\vdash_{\Sigma} \varphi A \rightarrow \psi B \quad \Leftrightarrow \quad \vdash_{\Sigma} \psi^{*} B \rightarrow \varphi^{*} A$

  where φ and ψ are any modalities (sequences of ¬ , □ and ◇) and φ* and ψ* are obtained from by interchanging □ and ◇.

- Main normal systems:

  — *K* = the smallest system

  — The main extensions are obtained by adding one or more of the following:

  D $\Box A \rightarrow \Diamond A$
  T $\Box A \rightarrow A$
  B $A \rightarrow \Box\Diamond A$
  4 $\Box A \rightarrow \Box A$
  5 $\Diamond A \rightarrow \Box\Diamond A$

  — Naming conventions:

  $KS_1 \ldots S_n$

  is the (smallest) extension of K obtained by taking the schemas $\boldsymbol{S_1} \ldots \boldsymbol{S_n}$ as axioms. (The order of the S_i does not matter.)

  — E.g., *KT5* is the smallest system of modal logic obtained by adding T and 5—etc.

- Facts:

  — There are $\mathbf{2^5}$ =**32** possible combinations

  — Only **15** of these are distinct

  — General picture: Chellas <u>figure 4.1</u> on p. 132.

- Example:

  KTD = KT

  Proof: Obviously

  KT   KTD.

  So we only show that

  KTD   KT.

  To this end it is sufficient to show that every instance of *D* is a theorem of *KT*

1.	$\vdash_{KT} \Box A \to A$	T
2.	$\vdash_{KT} A \to \Diamond A$	duality principle
3.	$\vdash_{KT} \Box A \to \Diamond A$	1,4 PL

- Other examples:

  KT5 = KTD5 = KTB5 = KT45 = KTDB5 = KTD45 = KTB45 = KTDB45   (This is **S5**)

  — Every instance of *D* is a theorem of *KT5*: obvious from above

  — Every instance of *B* is a theorem of *KT5*:

1.	$\vdash_{KT5} \Diamond A \to \Box\Diamond A$	5
2.	$\vdash_{KT5} A \to \Diamond A$	dual of T
3.	$\vdash_{KT} A \to \Box\Diamond A$	1,2 PL

  — Every instance of *4* is a theorem of *KT5*:

1.	$\vdash_{KT5} \Diamond A \to \Box\Diamond A$	5
2.	$\vdash_{KT5} \Diamond\Box A \to \Box A$	5◊ (duality principle)
3.	$\vdash_{KT5} \Box\Diamond\Box A \to \Box\Box A$	2, RM
4.	$\vdash_{KT5} \Box A \to \Box\Diamond\Box A$	B (which is a theorem of KT5)
5.	$\vdash_{KT5} \Box A \to \Box\Box A$	3,4, PL

4. Reduction laws for modalities

- Definition: two modalities φ and ψ are equivalent (in system Σ) iff for all sentences

  $\vdash_{\Sigma} \varphi A \leftrightarrow \psi A$

- Example: in *KT5* there are at most 6 distinct modalities: $A, \Diamond A, \Box A, \neg A, \neg\Diamond A, \neg\Box A$.

a)	$\vdash_{KT5} \Box\Box A \leftrightarrow \Box A$	1.	$\vdash_{KT5} \Box\Box A \rightarrow \Box A$	T
		2.	$\vdash_{KT5} \Box A \rightarrow \Box\Box A$	4
		3.	$\vdash_{KT5} \Box A \leftrightarrow \Box\Box A$	1,2, PL
b)	$\vdash_{KT5} \Diamond\Diamond A \leftrightarrow \Diamond A$	1.	$\vdash_{KT5} \Diamond\Diamond A \rightarrow \Diamond A$	4$\Diamond$
		2.	$\vdash_{KT5} \Diamond A \rightarrow \Diamond\Diamond A$	T$\Diamond$
		3.	$\vdash_{KT5} \Diamond\Diamond A \leftrightarrow \Diamond A$	1,2, PL
c)	$\vdash_{KT5} \Box\Diamond A \leftrightarrow \Diamond A$	1.	$\vdash_{KT5} \Box\Diamond A \rightarrow \Diamond A$	T
		2.	$\vdash_{KT5} \Diamond A \rightarrow \Box\Diamond A$	5
		3.	$\vdash_{KT5} \Box\Diamond A \leftrightarrow \Diamond A$	1,2, PL
d)	$\vdash_{KT5} \Diamond\Box A \leftrightarrow \Box A$	1.	$\vdash_{KT5} \Box A \rightarrow \Diamond\Box A$	T$\Diamond$
		2.	$\vdash_{KT5} \Diamond\Box A \rightarrow \Box A$	5$\Diamond$
		3.	$\vdash_{KT5} \Diamond\Box A \leftrightarrow \Box A$	1,2, PL

- Example:

$\vdash_{KT5} \Diamond\Box\Box\neg\Box\Diamond\neg\Box\Diamond\Box A$	$\leftrightarrow$	$\vdash_{KT5} \Diamond\Box\Box\neg\Box\neg\Box\neg\neg\Box\Diamond\Box A$	Df$\Diamond$
	$\leftrightarrow$	$\vdash_{KT5} \Diamond\Box\Box\neg\Box\neg\Box\Box\Diamond\Box A$	PL + REP
	$\leftrightarrow$	$\vdash_{KT5} \Diamond\Box\Box\Diamond\Box\Box\Diamond\Box A$	Df$\Diamond$
	$\leftrightarrow$	$\vdash_{KT5} \Box\Box\Diamond\Box\Box\Diamond\Box A$	*d)* above
	$\leftrightarrow$	$\vdash_{KT5} \Box\Diamond\Box\Box\Diamond\Box A$	*a)* above
	$\leftrightarrow$	$\vdash_{KT5} \Diamond\Box\Box\Diamond\Box A$	*c)* above
	$\leftrightarrow$	$\vdash_{KT5} \Box\Box\Diamond\Box A$	*d)* above
	$\leftrightarrow$	$\vdash_{KT5} \Box\Diamond\Box A$	*a)* above
	$\leftrightarrow$	$\vdash_{KT5} \Diamond\Box A$	*c)* above
	$\leftrightarrow$	$\vdash_{KT5} \Box A$	*d)* above

- In fact, you can just drop all modalities except for the last (plus negation, if necessary)

- Remarks:

1) These reduction laws fix an <u>upper bound</u>; a <u>lower bound</u> (to the effect that there are no further reduction laws) follows from completeness.

2) Only 7 of the 15 basic systems in the picture have <u>finitely many</u> distinct modalities:

KT4 K5 KD5 K45 KB4 KD45 KT5

3) Two systems may have the same modalities, but differ with respect to the patterns of implication among them (though not the other way around).

— e.g. KT5 and KD45 have the same six modalities, but T is only provable in KT5.