MODAL LOGIC 1.2 — SENTENTIAL MODAL LOGIC **LOA 28/4/3**

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1. Syntactic Preliminaries: the Modal Language

◆ Vocabulary:

— atomic formulas: $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \ldots$
— connectives: , T, ¬ , , , →, ↔ , □, ◇.
— metavariables: A, B, C, ...

◆ Grammar:

— Straightforward.
— Only be careful to distinguish necessity of the <u>consequence</u> vs necessity of the <u>consequent</u>.

1) $\Box(A \to B)$
2) $A \to \Box B$

Obviously different:

$\Box(\mathbb{P} \to \mathbb{P})$ ($\mathbb{P} \to \mathbb{P}$ is a tautology)
$\mathbb{P} \to \Box\mathbb{P}$

Often ambiguous in English

If I have no money, then I can't buy a new computer this probably corresponds to 1)
If I am a man, then I can't be a number this probably corresponds to 2)

2. Semantics

◆ <u>Extensional</u> models are (intuitively) <u>possible worlds</u>

— Each model α is a way of partitioning the atomic sentences into true and false:

$\alpha(\mathbb{P}_i)$ {T,F} for all *i*

or simply

α $\{\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \ldots\}$

— This induces an corresponding assignment of values to all sentences—a valuation:

$$V_\alpha(A) \rightarrow \{T,F\} \text{ for all sentences } A$$

or equivalently

$$\models_\alpha A \text{ iff } A \text{ is true (holds, etc.) relative to } \alpha.$$

— This is done recursively:

$$\models_\alpha \mathbb{P}_i \quad \text{iff} \quad \mathbb{P}_i \quad \alpha$$
$$\models_\alpha \neg A \quad \text{iff} \quad \text{not} \models_\alpha A$$
$$\models_\alpha A \quad B \quad \text{iff} \quad \models_\alpha A \text{ and} \models_\alpha B$$
$$\vdots \qquad \vdots \qquad \vdots$$

◆ How do we specify truth conditions for modal formulas (given that □, ◇ are not truth-functional)?

1) Carnap in *Meaning and Necessity*:

$$\models_\alpha \Box A \quad \text{iff} \quad \models_\beta A \text{ for every model } \beta$$
$$\models_\alpha \Diamond A \quad \text{iff} \quad \models_\beta A \text{ for some model } \beta$$

— This is too strict: it equates worlds with models, hence necessity and logical validity.

2) Leibniz (on modern readings):

— A model is not just a possible world, but a <u>collection of possible worlds</u>.

— Hence a model is a <u>collection of extensional models</u>.

$$\mathscr{M} = \{\alpha, \beta, \gamma, ...\}$$

— Then we could say e.g.

$$\models^{\mathscr{M}}_\alpha \mathbb{P}_i \quad \text{iff} \quad \mathbb{P}_i \quad \alpha$$
$$\models^{\mathscr{M}}_\alpha \neg A \quad \text{iff} \quad \text{not} \models^{\mathscr{M}}_\alpha A$$
$$\models^{\mathscr{M}}_\alpha A \quad B \quad \text{iff} \quad \models^{\mathscr{M}}_\alpha A \text{ and} \models^{\mathscr{M}}_\alpha B \qquad .$$
$$\models^{\mathscr{M}}_\alpha \Box A \quad \text{iff} \quad \models^{\mathscr{M}}_\beta A \text{ for every } \beta \quad \mathscr{M}$$
$$\models^{\mathscr{M}}_\alpha \Diamond A \quad \text{iff} \quad \models^{\mathscr{M}}_\beta A \text{ for some } \beta \quad \mathscr{M}$$

— This account still requires that we specify <u>what</u> possible worlds are (functions or sets of sentences, etc.), but otherwise OK. It can be generalized as follows:

◆ **First idea** (Leibniz's simplified models)

— Instead of a <u>set of functions</u>, a model becomes a <u>set with a function</u>

- take worlds as unanalyzed entities (points)
- ask the model to associate each sentence with the worlds in which it is true

— Formally: $\mathscr{M} = \; W, P \;$, where

- $W \neq \varnothing$ (the possible worlds)
- P a sequence $P_0, P_1, P_2, \ldots \quad W$ associating with each i a set of worlds (those in which $\mathbb{P}_i$ holds)
- Intuitively: P_i = the proposition expressed by $\mathbb{P}_i$

— Truth conditions:

$$\models^{\mathscr{M}}_{\alpha} \mathbb{P}_i \quad \text{iff} \quad \alpha \quad P_i$$
$$\vdots \qquad\qquad \vdots \quad \vdots$$
$$\models^{\mathscr{M}}_{\alpha} \Box A \quad \text{iff} \quad \models^{\mathscr{M}}_{\beta} A \text{ for every } \beta \quad W$$
$$\models^{\mathscr{M}}_{\alpha} \Diamond A \quad \text{iff} \quad \models^{\mathscr{M}}_{\beta} A \text{ for some } \beta \quad W$$

— Notes:

- P_i may be empty
- $\bigcup_i P_i$ may not add up to W

◆ **Second idea** (Kripke's <u>standard</u> models): generalize Leibnizian models by adding an accessibility relation:

— Model $\mathscr{M} = \; W, R, P \;$, where

- W and P as before
- $R \quad W \times W$

— Truth conditions:

$$\models^{\mathscr{M}}_{\alpha} \mathbb{P}_i \quad \text{iff} \quad \alpha \quad P_i$$
$$\vdots \qquad\qquad \vdots \quad \vdots$$
$$\models^{\mathscr{M}}_{\alpha} \Box A \quad \text{iff} \quad \models^{\mathscr{M}}_{\beta} A \text{ for all } \beta \quad W \text{ such that } \alpha R \beta$$
$$\models^{\mathscr{M}}_{\alpha} \Diamond A \quad \text{iff} \quad \models^{\mathscr{M}}_{\beta} A \text{ for some } \beta \quad W \text{ such that } \alpha R \beta$$

— NB: If R is an <u>equivalence relation</u>, this is equivalent to the first account.

◆ **Third idea** (Montague's <u>minimal</u> models): Modal notions should <u>not</u> be understood in terms of truth in every/some world, but treated as primitive: some sentences express necessay/possible propositions, others do not.

— Model $\mathscr{M} = \; W, N, P \;$, where

- $N : W \rightarrow \qquad W$ associates each world with the propositions that are necessary at that world

  W = sets of worlds = propositions

  W= set of propositions

- Intuitively: N_α = the propositions that are necessary at α

— Truth conditions:

$\models^{\mathscr{M}}_{\alpha} \mathbb{P}_i$ iff α P_i
⋮ ⋮ ⋮
$\models^{\mathscr{M}}_{\alpha} \Box A$ iff $\{\beta \quad W: \models^{\mathscr{M}}_{\beta} A\} \quad N_\alpha$ = iff A expresses a necessary proposition at α
$\models^{\mathscr{M}}_{\alpha} \Diamond A$ iff $\{\beta \quad W: \text{not} \models^{\mathscr{M}}_{\beta} A\} \quad N_\alpha$ = iff A does not express an impossible proposition at α

— Notation: $||A||^{\mathscr{M}}$ for $\{\beta \quad W: \models^{\mathscr{M}}_{\beta} A\}$ (the proposition expressed by A in $\mathscr{M}$)

3. Examples

◆ Definitions

— A is valid/true in $\mathscr{M}$	$\models^{\mathscr{M}} A$	iff	$\models^{\mathscr{M}}_{\alpha} A$ for every α W
— A is valid in C	$\models_{C} A$	iff	$\models^{\mathscr{M}} A$ for every $\mathscr{M}$ C
— A is valid	$\models A$	iff	$\models_{C} A$ for very C

◆ Some principles that are valid in the semantics based on Leibnizian models:

D $\Box A \rightarrow \Diamond A$
T $\Box A \rightarrow A$
B $A \rightarrow \Box\Diamond A$
4 $\Box A \rightarrow \Box\Box A$
5 $\Diamond A \rightarrow \Box\Diamond A$
G $\Diamond\Box A \rightarrow \Box\Diamond A$

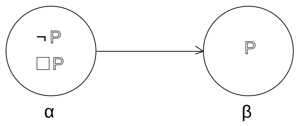
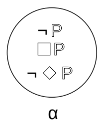
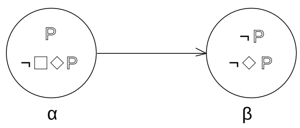
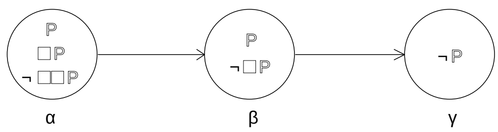
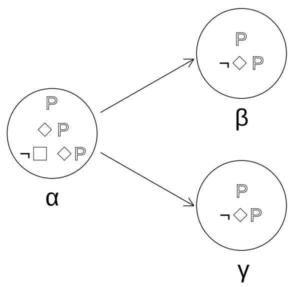
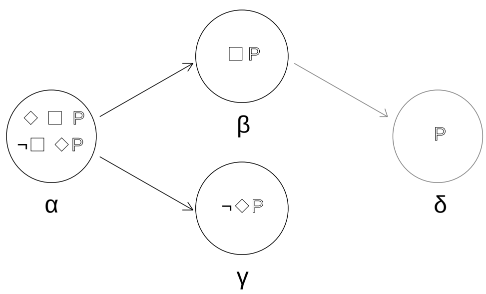
K $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
Df◇ $\Diamond A \leftrightarrow \neg\Box\neg A$
Df□ $\Box A \leftrightarrow \neg\Diamond\neg A$

RN $\dfrac{\models_{C} A}{\models_{C} \Box A}$

RE $\dfrac{\models_{C} A \leftrightarrow B}{\models_{C} \Box A \leftrightarrow \Box B}$

RK $\dfrac{\models_{C} (A_1 \quad \ldots \quad A_n) \rightarrow A}{\models_{C} (\Box A_1 \quad \ldots \quad \Box A_n) \rightarrow \Box A}$

◆ Comparison with the other semantics:

schema	*standard models*	*minimal models*
T $\Box A \rightarrow A$	valid iff *reflexive*: αRα 	valid iff α ∥A∥$^{\mathcal{M}}$ whenever ∥A∥$^{\mathcal{M}}$ N_α
D $\Box A \rightarrow \Diamond A$	valid iff *serial*: α β(αRβ) 	valid iff –∥A∥$^{\mathcal{M}}$ N_α whenever ∥A∥$^{\mathcal{M}}$ N_α
B $A \rightarrow \Box\Diamond A$	valid iff *symmetric*: αRβ βRα 	(b) p. 224 Chellas
4 $\Box A \rightarrow \Box\Box A$	valid iff *transitive*: αRβ and βRγ αRγ 	(iv) p. 224
5 $\Diamond A \rightarrow \Box\Diamond A$	valid iff *euclidean*: αRβ and αRγ βRγ 	(v) p. 224
G $\Diamond\Box A \rightarrow \Box\Diamond A$	valid iff *incestual*: αRβ & αRγ δ(βRδ & γRδ) 	(g) p. 225

schema	*standard models*	*minimal models*
K $\Box(A \to B) \to (\Box A \to \Box B)$	valid	valid iff $\|B\|^{\mathcal{M}}\ \ N_\alpha$ whenever $\|A \to B\|^{\mathcal{M}}, \|A\|^{\mathcal{M}}\ \ N_\alpha$
Df◇ $\Diamond A \leftrightarrow \neg\Box\neg A$	valid	valid
Df□ $\Box A \leftrightarrow \neg\Diamond\neg A$	valid	valid
RN $\dfrac{\models_C A}{\models_C \Box A}$	valid	valid iff $W\ \ N_\alpha$ (for all α in all $\mathcal{M}$ in C)
RE $\dfrac{\models_C A \leftrightarrow B}{\models_C \Box A \leftrightarrow \Box B}$	valid	valid
RK $\dfrac{\models_C (A_1\ \ \dots\ \ A_n) \to A}{\models_C (\Box A_1\ \ \dots\ \ \Box A_n) \to \Box A}$	valid	valid iff $\|A\|^{\mathcal{M}}\ \ N_\alpha$ whenever $\|A_1\|^{\mathcal{M}}, \dots, \|A_n\|^{\mathcal{M}}\ \ N_\alpha$

◆ Example of proof for the "if" part: Scheme 5 is valid in the class of all euclidean standard models:

1. Assume $\models^{\mathcal{M}}_{\alpha} \Diamond A$
2. Then $\models^{\mathcal{M}}_{\beta} A$ for some β W such that αRβ
3. Suppose αRγ
4. Then γRβ by euclideanness
5. So, for any γ such that αRγ there exists β such that γRβ and $\models^{\mathcal{M}}_{\beta} A$
6. So, for any γ such that αRγ, $\models^{\mathcal{M}}_{\gamma} \Diamond A$
7. Thus $\models^{\mathcal{M}}_{\alpha} \Box\Diamond A$
8. By 1–7, if $\models^{\mathcal{M}}_{\alpha} \Diamond A$ then $\models^{\mathcal{M}}_{\alpha} \Box\Diamond A$
9. Hence $\models^{\mathcal{M}}_{\alpha} \Diamond A \to \Box\Diamond A$

4. General comparison

◆ DEFINITION: Two structures $\mathcal{M} = \ W, \dots, P\ $ and $\mathcal{M}' = \ W', \dots, P'\ $ are pointwise equivalent iff there is a one-one map $f : W \to W'$ such that, for every sentence A and every α W

$$\models^{\mathcal{M}}_{\alpha} A \quad \text{iff} \quad \models^{\mathcal{M}'}_{f(\alpha)} A$$

◆ <u>Fact 1</u>: Every simplified model $\mathcal{M} = \; W, P$ is pointwise equivalent to a standard model, namely to the model $\mathcal{M}' = \; W, R, P$ where $R = W \times W$.

<u>Proof</u>: straightforward inductive argument, taking $f(\alpha) = \alpha$.

1. Base: $\models^{\mathcal{M}}_{\alpha} \mathbb{P}_i$ iff $\alpha \quad P_i$
   iff $\models^{\mathcal{M}'}_{\alpha} \mathbb{P}_i$
   i.e. $\models^{\mathcal{M}'}_{f(\alpha)} \mathbb{P}_i$

2. Truth-functional connectives: obvious

3. Modal connectives:

$\models^{\mathcal{M}}_{\alpha} \Box A$ iff $\models^{\mathcal{M}}_{\beta} A$ for all $\beta \quad W$
iff $\models^{\mathcal{M}'}_{\beta} A$ for all $\beta \quad W$ (by I.H.)
iff $\models^{\mathcal{M}'}_{\beta} A$ for all $\beta \quad W$ such that $\alpha R \beta$
iff $\models^{\mathcal{M}'}_{\alpha} \Box A$
i.e. $\models^{\mathcal{M}'}_{f(\alpha)} \Box A$

◆ <u>Fact 2</u>: Every standard model $\mathcal{M}^s = \; W^s, R, P^s$ is pointwise equivalent to a minimal model $\mathcal{M}^m = \; W^m, N, P^m$, where X $\quad N_\alpha$ iff X contains all R-accessible worlds, i.e., iff $\{\beta \quad W: \alpha R \beta\} \quad$ X.

(Intuitively: the propositions necessary at α are those that include the set of all worlds accessible from α)

<u>*Proof*</u>: we set $f(\alpha) = \alpha$ and prove by induction that, for every sentence *A:*

for every $\alpha \quad W$: $\models^{\mathcal{M}^s}_{\alpha} A$ iff $\models^{\mathcal{M}^m}_{\alpha} A$

Again, the only interesting case is modal sentences:

$\models^{\mathcal{M}^s}_{\alpha} \Box A$ iff $\models^{\mathcal{M}^s}_{\beta} A$ for all $\beta \quad W$ s.t. $\alpha R \beta$
iff $\{\beta \quad W: \alpha R \beta\} \quad \{\beta \quad W: \models^{\mathcal{M}^s}_{\beta} A\}$
iff $\{\beta \quad W: \alpha R \beta\} \quad \{\beta \quad W: \models^{\mathcal{M}^m}_{\beta} A\}$ by I.H.
iff $\{\beta \quad W: \models^{\mathcal{M}^m}_{\beta} A\} \quad N_\alpha$ by def. of $\mathcal{M}^m$
iff $\models^{\mathcal{M}^m}_{\alpha} \Box A$ by recursive clause for $\Box$