A Pointless Theory of Space Based On Strong Connection and Congruence

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Abstract

We present a logical theory of space where only tridimensional regions are assumed in the domain. Three distinct primitives are used to describe their mereological, topological and morphological properties: mereology is described by a parthood relation satisfying the axioms of Closed Extensional Mereology; topology is described by means of a "simple region" predicate, by which a relation of “strong connection” between regions having at least a surface in common is defined; morphology is described by means of a "congruence" primitive, whose axioms exploit Tarski's analogy between points and spheres.

1 INTRODUCTION

Various logical theories aimed at the representation of commonsense spatial knowledge have been proposed in the AI community in recent years. In the spirit of (Hayes 1985a), the goal has been that of establishing the logical basis of a “geometry of commonsense”, intended to be used for tasks as disparate as robot navigation or natural language understanding. Besides specific proposals focused on particular aspects, like (Hayes 1985b, Davis 1993, Shanahan 1995), the most general frameworks have been perhaps those based on a combination of mereological and topological notions, which appeared in AI after the publication of Clarke’s mereotopological theory (Clarke 1981). Among these, particular relevance for AI purposes has the so-called RCC theory (Randell and Cohn 1992, Gotts 1994, Bennett 1995, Gotts *et al.* 1996), which has been recently joined by other proposals originating in the area of philosophy and linguistics (Aurnague and Vieu 1993, Asher and Vieu 1995, Eschenbach and Heydrich 1995, Casati and Varzi 1996, Smith 1996). These approaches differ in the primitives adopted and in the ontological assumptions about the domain. They all have in common however the use of the tools of so-called “formal ontology” (Guarino 1995) for the representation of commonsense reality: specifically, mereology and topology. Readers can refer to (Simons 1987) for a general overview of mereology, and to (Varzi 1994, Varzi 1996b) for a systematic account of the subtle relations between mereology and topology.

We can distinguish four main aspects which are useful to compare and evaluate ontological theories of space, namely: i) the assumptions related to the ontological status of spatial entities (conceived of as existing either only on its own or rather only relative to physical objects); ii) the presence/absence of lower dimension entities like *boundaries* (surfaces, lines or points) in the domain; iii) the primitives used to express the relevant relations; iv) the degree of characterization of the intended models. In this paper, we are presenting a logical theory whose main contributions are discussed and motivated in the light of these points.

In section 2 we address the first two points, discussing our basic ontological assumptions about space. We adopt the classical conception of absolute space, excluding lower dimension entities from the domain; our individuals are therefore three-dimensional regions.

The discussion of points iii and iv represents the body of the paper. We assume three distinct levels of description of space: the *mereological level*, the *topological level*, and the *morphological level*. A specific primitive is adopted for each of these levels.

In fact, since our domain is limited to spatial entities only, we could use the connection (‘C’) primitive to define parthood (as done by Clarke), eliminating of a primitive. We prefer however to keep the mereological and topological levels separate because of reasons of cognitive clarity and immediateness, adopting a general parthood primitive supplemented with a specific topological primitive. Our choice differs therefore from the approaches inspired by Clarke's work, since we don't take connection (‘C’) as a primitive: rather, we introduce a unary *simple region* (‘SR’) primitive, where a simple region (shortly, *s-region*), is a region "all in one piece", i.e. a *strongly* self-connected region, such that any two halves of it always have a surface in common (*strong connection* or *surface connection,* briefly *s-connection*). In other words, an s-region, in everyday intuition, can be occupied by a single thing. The reasons of this choice come from the desire to make possible a simple and natural interpretation of the very notion of (external) connection, which is far from being totally clear, especially for what concerns the distinctions between point-, line- and surface-connection.

A distinguishing feature of the theory presented here is the emphasis given to the morphological level. We take the *congruence relation* between regions as a primitive, proposing an axiomatization which is constructed upon the reconstruction of Euclidean geometry in terms of spheres developed by Tarski (Tarski 1956). Spheres, in turn, are defined in terms of parthood, s-regions and congruence. The resulting theory turns out to be extremely powerful, allowing us to rigorously distinguish between point-, line-, and surface-connection, and to define notions such as convexity and granularity, which we plan to exploit in a research project aimed at the logical modelling of mechanical assemblies (Borgo *et al.* 1996).

For each of the three levels introduced above, we propose an axiomatization aimed at achieving a good characterization of our intended models. This last point marks a major feature of our approach: our theory is an ontological theory, in the sense discussed in (Guarino and Giaretta 1995). It is a *rich* theory in terms of axioms and definitions (in the spirit of (Hayes 1985)), since its main purpose is to *convey meaning*, in such a way as to characterize more precisely the ontological assumptions underlying the primitives adopted, restricting their possible interpretations. The high expressiveness of the language (full first order logic) makes the theory not suitable for direct implementation in a reasoning system, while making it possible however to adopt its primitives and definitions as the basis of a logical vocabulary suitable to be *shared* among different applications (Guarino *et al.* 1994).

The rest of this paper develops as follows. In section 2 we introduce in more detail our basic ontological assumptions; sections 3, 4 and 5 are devoted to the presentation of our mereological, topological and morphological axioms, respectively; finally, in section 6 we make an overall assessment of the theory with respect to the current literature, and we briefly discuss its potential applications.

2 BASIC ONTOLOGICAL ASSUMPTIONS

We have adopted here the classical Newtonian conception of space, according to which space exists independently of the entities that can occupy it. In this view, space is like a container where physical objects can be located and can move. In other words, space is intended as a *substrate,* since its existence is a necessary condition for physical objects to exist (Borgo *et al.* 1996). Other proposals more oriented towards natural language applications, like (Aurnague and Vieu 1993), adopt the Leibnizian conception, assuming space to be strictly dependent on the relations holding between physical objects. In this second view, it is not necessary to admit the existence of spatial entities like regions, and, in any case, the existence of such entities depends directly on the objects themselves.
Although both of these claims are tenable, the first position seems to be much easier for practical applications. For instance, in the absolute conception, the movement of an object could be described quite simply as the sequence of regions where the object is located, whereas otherwise we would have to consider the changes of the *relevant* relations between our object and the other objects in the system.

A further crucial assumption is related to the exclusion of spatial entities like surfaces, lines, and points from our domain. They are basic ingredients of ordinary geometry, but their ontological status in our everyday interaction with the world is debatable. In fact, our experience of space is strictly related to physical three-dimensional objects (our human body is one of them). We move these objects around to perceive space and to recognise the "outside" world. From this naive position, points are not considered as inhabitants of this world, but rather results of complex abstract reasonings, so they could be defined, if really necessary, as higher-order entities. In this spirit, various proposals for an axiomatization of *pointless geometry* have been made in the past (Lobachevskij 1835, Whitehead 1929, Grzegorczyk 1960) (see (Gerla 1994) for an overview). Approaches based on Clarke's axiomatization, like (Asher and Vieu 1995, Gotts *et al.* 1996), don't admit points in their domain[1]. The RCC theory moves further, getting rid of the topological distinction between open and closed regions to avoid a number of problems related to the boundaries of physical objects. We have adopted this latter choice, assuming regular three-dimensional regions in a 3D Euclidean space as elements of our domain. The peculiar property exhibited by these regions is that they can be occupied by concrete physical objects. Indeed, although the domain assumed in the formalization below is limited to spatial regions only, all the ontological choices made in this paper are motivated by a more general view where space, matter and physical objects are considered (Borgo *et al.* 1996).

3 MEREOLOGICAL LEVEL

Let us present the axioms assumed for our primitive parthood relation, represented by means of the binary predicate 'P'. It is important to stress that our domain is limited here to spatial regions; in this restricted domain, parthood can be taken as equivalent to spatial inclusion, since the various problems bound to the ontological nature of the entities involved disappear: for instance, it makes no sense to distinguish an object in a hole (which is not part of the hole) from the region it occupies (which is part

[1] See however (Varzi 1996a) and (Smith 1996) for a different strategy, where the peculiar ontological status of *boundaries* is defended.

of the region occupied by the hole) (Casati and Varzi 1995).

We have adopted in the following a standard first-order language with identity. Throughout the paper, free variables appearing in formulas are intended to be universally quantified. The following definitions are built upon the parthood relation[1]:

D1. $\mathrm{PP}xy =_{df} \mathrm{P}xy \wedge \neg x{=}y$ *(Proper part)*

D2. $\mathrm{O}xy =_{df} \exists z(\mathrm{P}zx \wedge \mathrm{P}zy)$ *(Overlap)*

D3. $\mathrm{PO}xy =_{df} \mathrm{O}xy \wedge \neg \mathrm{P}xy \wedge \neg \mathrm{P}yx$ *(Proper overlap)*

D4. $x{+}y =_{df} \iota z \forall w(\mathrm{O}wz \leftrightarrow (\mathrm{O}wx \vee \mathrm{O}wy))$ *(Sum)*

D5. $x{-}y =_{df} \iota z \forall w(\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \neg \mathrm{O}wy))$ *(Difference)*

D6. $x{\times}y =_{df} \iota z \forall w(\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \mathrm{P}wy))$ *(Product)*

The ι operator appearing above is defined contextually *à la* Russel:

D7. $\psi[\iota x \phi] =_{df} \exists y(\forall x(\phi \leftrightarrow x{=}y) \wedge \psi[y])$

The following axioms – equivalent to those of Closed Extensional Mereology (Simons 1987, Varzi 1996b) – are introduced:

A1. $\mathrm{P}xx$

A2. $\mathrm{P}xy \wedge \mathrm{P}yx \rightarrow x{=}y$

A3. $\mathrm{P}xy \wedge \mathrm{P}yz \rightarrow \mathrm{P}xz$

A4. $\exists z(z{=}x{+}y)$

A5. $\neg \mathrm{P}xy \rightarrow \exists z(z{=}x{-}y)$

Notice that the so-called "fusion axiom" (allowing for infinite sums) typical of General Extensional Mereology is not assumed, since it would lead to inconsistencies in our mereo-topological framework (see below). Some simple theorems deriving from the axioms above are the following:

T1. $\neg \mathrm{P}xy \rightarrow \exists z(\mathrm{P}zx \wedge \neg \mathrm{O}zy)$ (A1; A5; D5)
(strong remainder principle)

T2. $\mathrm{O}xy \rightarrow \exists z(z{=}x{\times}y)$ (A4; A5; D6)
(existence of product)

4 TOPOLOGICAL LEVEL

As mentioned in the introduction, we have decided to adopt a unary primitive 'SR' instead of 'C'. To make clear this choice it is important to analize the relation between mereology and topology a little more. In the past, various attempts have been made to define topological notions in terms of parthood. In fact, the overlap relation defined by D2:

$\mathrm{O}xy =_{df} \exists z(\mathrm{P}zx \wedge \mathrm{P}zy)$

seems to resemble a notion akin to that of connection. The crucial point here is the ontological nature of the elements involved in the above definition. As pointed out in (Eschenbach and Heydrich 1995), if we have both points and regions in the domain we can restrict the arguments of the "overlap" relation to hold only for regions, by means of a primitive predicate 'R', which selects (regular) regions. Topological connection can be therefore defined as follows:

$\mathrm{C}xy =_{df} \mathrm{R}x \wedge \mathrm{R}y \wedge \mathrm{O}xy$ *(connection)*

In the case of external connection, this means that x and y have a part in common (i.e., they overlap), but this part is not a region (assuming regions as three-dimensional, it may be a point, a line, or a surface).

Notice that the primitive 'R' above isolates some entities in the domain which have a peculiar "topological" meaning. In our case, since the domain is *already* limited to regions, the definition above would collapse to ordinary overlap. We must then resort to some other strategy. An attempt in this sense has been made by Whitehead (Whitehead 1929), who tried to define connection as follows, by means of mereology only:

$\mathrm{C}xy =_{df} \exists z(\mathrm{O}zx \wedge \mathrm{O}zy \wedge \forall w(\mathrm{P}wz \rightarrow (\mathrm{O}wx \vee \mathrm{O}wy)))$

As shown in (Varzi 1994), it is easy to see however that the above definition fails if we allow z to be a disconnected region (Figure 1).

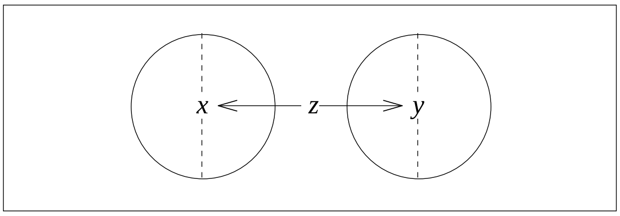

Figure 1. Whitehead's problem: x and y are not connected unless the overlaying piece z is itself connected. From (Casati and Varzi 1995).

In other words, we face here the main problem of mereology, namely the impossibility of characterizing the notion of a *whole*. The move we have made to solve this problem is simply the introduction of this notion at the topological level by means of the primitive 'SR' which denotes a *self-connected* regular region. This will allow us to define "connection" in terms of 'P' and 'SR' by means of a simple specialization of Whitehead's definition:

$$\mathrm{C}xy =_{df} \exists z(\mathrm{SR}z \wedge \mathrm{O}zx \wedge \mathrm{O}zy \wedge \forall w(\mathrm{P}wz \rightarrow (\mathrm{O}wx \vee \mathrm{O}wy)))$$

We see therefore that an extra primitive besides parthood is necessary to define connection. The *meaning* of the connection relation, however, remains still obscure unless the 'SR' primitive is suitably characterized. Our choice

[1] Throughout the paper, we mark definitions with 'D', axioms with 'A' and theorems with 'T'

has been to interpret ‘SR’ in a strong sense, as denoting s-regions rather than generic self-connected regions. Under this interpretation, the definition above refers to *strong connection* (*s-connection*). In our intuition, such a notion is bound to that of *physical connection*.. For an example of a *non* s-connected region, think of two spheres (say, two apples) touching each other in a point: no physical object (like a worm for instance, however thin) can pass from one to the other without exposing itself to the outside space (Figure 2). We shall say in this case that the two spheres are connected but not s-connected. Later, with the help of morphology, we will be able to distinguish also between point- and line-connection (*p-connection* and *l-connection*).

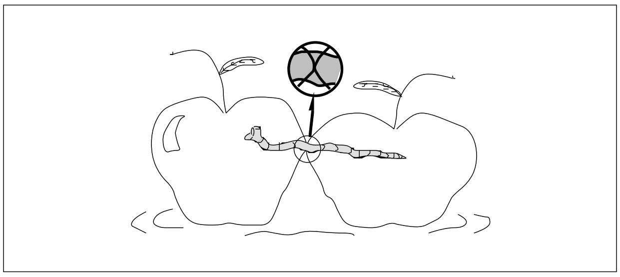

Figure 2. The regions occupied by the two apples are not s-connected. Their sum is not an s-region.

After this discussion of our *desiderata*, let us introduce the axioms and definitions which characterize our topological level, together with some interesting theorems.

Definitions:

D8. $\mathrm{IP}xy =_{\mathrm{df}} \mathrm{PP}xy \wedge \forall z((\mathrm{SR}z \wedge \mathrm{PO}zx) \rightarrow \mathrm{O}z(y\text{-}x))$ *(interior part)*

D9. $\mathrm{MCP}xy =_{\mathrm{df}} \mathrm{P}xy \wedge \mathrm{SR}x \wedge \neg\exists z(\mathrm{P}zy \wedge \mathrm{SR}z \wedge \mathrm{PP}xz)$ *(maximally connected part)*

D10. $\mathrm{SC}xy =_{\mathrm{df}} \exists uv(\mathrm{P}ux \wedge \mathrm{P}vy \wedge \mathrm{SR}(u+v))$ *(strong connection)*

Axioms:

A6. $(\mathrm{SR}x \wedge x=y+z) \rightarrow \exists u(\mathrm{SR}u \wedge \mathrm{O}uy \wedge \mathrm{O}uz \wedge \mathrm{IP}ux)$

A7. $\exists$y MCPyx

A8. $\exists y\ (\mathrm{SR}y \wedge \mathrm{IP}xy)$

Definition D8 deserves some comments. Notice first that, in our interpretation, the relation 'IP' turns out to be different from the usual relation 'NTPP' (non-tangential proper part), since a region *x* being l- or p-connected with a region external to *y* must be considered as an interior part of *y* (Figure 3). A further observation is related to the possibility of allowing the infinite mereological sum of the interior parts of a given region. This would lead to inconsistencies, since such a sum would coincide with the maximal interior part, while no maximal interior part can be admitted by T10 (see below). For this reason we have excluded infinite sums from our mereological framework. A similar problem has been noticed in (Randell and Cohn 1992).

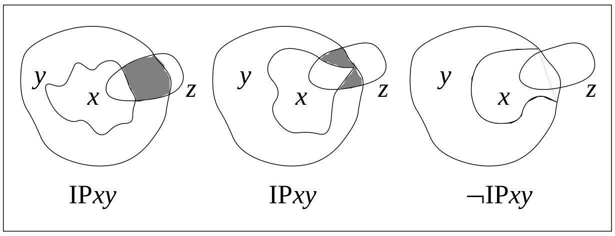

Figure 3. No external region can be *strongly* connected to interior parts.

Definition D9 introduces the notion of maximally connected part, and definition D10 is a compact rewriting of Whitehad’s amended definition reported above.

Axiom A6 refines the idea of strong connection between two halves of a s-region by adding the further requirement that the “connecting” region be an interior part of the resulting sum; in this way, it excludes regions which are not manifolds, like the sum of two tangent spheres, to be considered as a s-region. To see this, suppose *per absurdum* that the sum *x* of two p-connected regions *y* and *z* is a simple region (Figure 4). According to A6, we must find a “connecting” s-region *u* overlapping both *y* and *z*, which is at the same time an interior part of *x*. Now, the only possible candidates for *u* must be regions p-connected in the same point which connects *y* and *z*, but these can’t be interior parts of *x* since there will be other regions (like *w*) external to *x* touching *u* in the same point. An analogous argument applies to the case of l-connected regions.

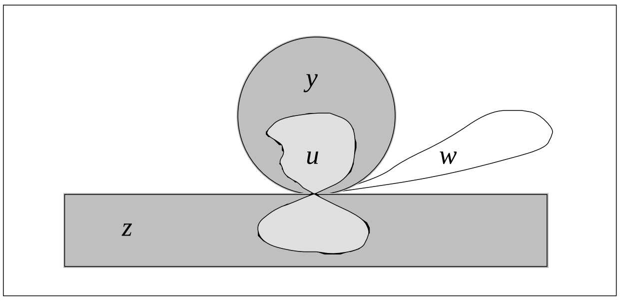

Figure 4. Why *y*+*z* cannot be an s-region.

Axioms A7 and A8 further determine the structure of our spatial domain. Through A7, every region has some maximally connected part; A8 states that any region is part of some simple region, and therefore the notions of universe and complement are not defined. Among other things, this axiom prevents 'SR' from being interpreted as a "local" property, like for example 'being red'.

Let us now present some theorems which follow from our axiomatization:

T3. $\mathrm{IP}xy \wedge \mathrm{P}yz \rightarrow \mathrm{IP}xz$ (A3; D8)

T4. $\mathrm{P}xy \wedge \mathrm{IP}yz \rightarrow \mathrm{IP}xz$ (A3; D8)

T5. $\mathrm{IP}xy \wedge \mathrm{IP}xz \rightarrow \mathrm{IP}x(y\times z)$ (D6; D8)

T6. $\mathrm{IP}xz \wedge \mathrm{IP}yz \rightarrow \mathrm{IP}(x+y)z$ (D4; D8)

T7. $\mathrm{SR}x \leftrightarrow \mathrm{MCP}xx$ (D9)

T8. $\neg\mathrm{SR}x \rightarrow \exists y(\mathrm{MCP}yx \wedge \mathrm{PP}yx)$ (A7; T7; D9)

T9. $\exists y(\mathrm{SR}y \wedge \mathrm{IP}yx)$ (A6; T3; T8; D9)

T10. $\mathrm{IP}xy \rightarrow \exists z(\mathrm{IP}zy \wedge \mathrm{PP}xz)$ (A4; A5; T3; T6; T9)

T11. $\mathrm{SR}x \rightarrow \forall yz(y+z=x \rightarrow \mathrm{SC}yz)$ (A6; D10)

T12. $\exists y(\mathrm{SC}xy \wedge \neg\mathrm{O}xy)$ (A5; A8)

T13. $\mathrm{SC}xx$ (A8; D10)

T14. $\mathrm{SC}xy \rightarrow \mathrm{SC}yx$ (D10)

T15. $\mathrm{P}xy \leftrightarrow \forall z(\mathrm{SC}zx \rightarrow \mathrm{SC}zy)$ (A3; A5; T1; T3; T9)

T16. $x=y \leftrightarrow \forall z(\mathrm{SC}zx \leftrightarrow \mathrm{SC}zy)$ (A2; T15)

Theorems T3-T6 establish some desirable properties of the relation 'IP' with respect to mereology, which are actually taken as axioms in (Smith 1996), where the 'IP' relation is adopted as a primitive; T7 shows that an s-region is maximally self-connected, while T8 shows that given a non-simple region it has a maximally connected proper part. T9 shows that our theory is not atomic, and that any region has an interior part which is an s-region. T10 shows that our space is dense, in the sense that, given a region *x* internal to *y*, there always exists another region *z* properly including *x* and still internal to *y*[1]. T11 shows that two halves of an s-region are always s-connected, while T12 shows that our notion of strong connection is actually different from overlap. Finally, T13-T14 show the reflexivity and the symmetry of s-connection, while T15-T16 establish the relationship of s-connection with parthood and identity, respectively.

5 MORPHOLOGICAL LEVEL

Besides mereological and topological relations, the possibility of expressing morphological features is a crucial aspect for any commonsense theory of space. A "convex hull" primitive has been used with some success within the RCC theory, but it seems too weak for our purpose. A ternary alignment relation has been used in (Aurnague and Vieu 1993), but it commits to a (higher-order) notion of point. We have opted here for a different morphological primitive, the most intuitive we can think of: the *congruence relation* between regions, designated by 'CG', meaning that CGxy holds if x and y have the same shape and size. In the case of classical geometry based on points, segments and angles, this relation was first axiomatized in (Hilbert 1902), with various simplifications thereafter. In our restricted domain, a "direct" axiomatization of the notion of congruence between regions seemed to be quite complicated to us, since we are not aware of any similar approach. We have chosen therefore to take advantage of Hilbert's work, by exploiting a correpondence between points and spheres. This correspondence has been brilliantly pursued in (Tarski 1956), where a (second order) axiomatic theory taking spheres and parts as primitives has been shown as equivalent to classical geometry. Our strategy to axiomatize 'CG' is the following:

1. define a sphere in terms of 'P', 'SR' and 'CG';

2. adopt Tarski's definitions related to spheres;

3. define a notion of alignment for three spheres;

4. formulate axioms for the congruence between binary sums of spheres (s-segments) by mimicking the standard axioms for congruence between segments, exploiting the analogy between points and spheres and the above definition of alignement for spheres

5. constrain the congruence between arbitrary regions in terms of congruence between binary sums of spheres.

6. add further axioms in order to constrain the congruence between spheres, and introduce suitable existential assumptions.

The result will be a *first order* theory of congruence between regions: with respect to Tarski's work, we *reconstruct* the congruence axioms in terms of spheres, rather than establishing a formal connection between points and classes of concentrical spheres (going therefore to second order). Moreover, we do not take spheres as a primitive, since we define them in terms of general mereological, topological, and morphological primitives.

5.1 DEFINING SPHERES

The crucial step in our axiomatization of congruence is the definition of a sphere, that makes it possible to embed Tarski's mereo-morphological theory within our framework:

D11. $\mathrm{SPH}x =_{\mathrm{df}} \mathrm{SR}x \wedge \forall y(\mathrm{CG}xy \wedge \mathrm{PO}xy \rightarrow \mathrm{SR}(x\text{-}y))$

A region *x* is a spere if and only if it is a simple region and it cannot be disconnected by another congruent sphere. It is easy to see that only spherical regions satisfy the above definition, which reflects the intrinsic symmetry of spheres (Figure 5), provided that enough regions congruent to the one given exist.

[1] This is a weak notion of density. We cannot prove a stronger notion as: $\mathrm{IP}xy \rightarrow \exists z(\mathrm{IP}zy \wedge \mathrm{IP}xz)$

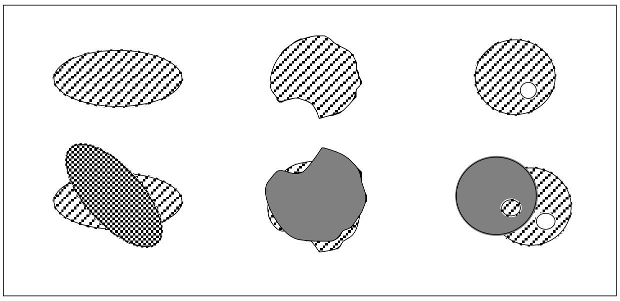

Figure 5. These regions are not spheres.

5.2 TARSKI'S DEFINITIONS

Let us now introduce Tarski's definitions regarding spheres. For the sake of conciseness, we assume that all variables appearing in D12-D21 below are restricted to the class of spheres.

D12. $\mathrm{ET}xy =_{\mathrm{df}} \neg \mathrm{O}xy \wedge ((\neg \mathrm{O}uy \wedge \neg \mathrm{O}vy \wedge \mathrm{P}xu \wedge \mathrm{P}xv) \rightarrow (\mathrm{P}uv \vee \mathrm{P}vu))$
(x is externally tangent to y)

D13. $\mathrm{IT}xy =_{\mathrm{df}} \mathrm{PP}xy \wedge ((\mathrm{P}uy \wedge \mathrm{P}vy \wedge \mathrm{P}xu \wedge \mathrm{P}xv) \rightarrow (\mathrm{P}uv \vee \mathrm{P}vu))$
(x is internally tangent to y)

D14. $\mathrm{ED}xyz =_{\mathrm{df}} \mathrm{ET}xz \wedge \mathrm{ET}yz \wedge (\neg \mathrm{O}uz \wedge \neg \mathrm{O}vz \wedge \mathrm{P}xu \wedge \mathrm{P}yv \rightarrow \neg \mathrm{O}uv)$
(*x and y are externally diametrical wrt z)*

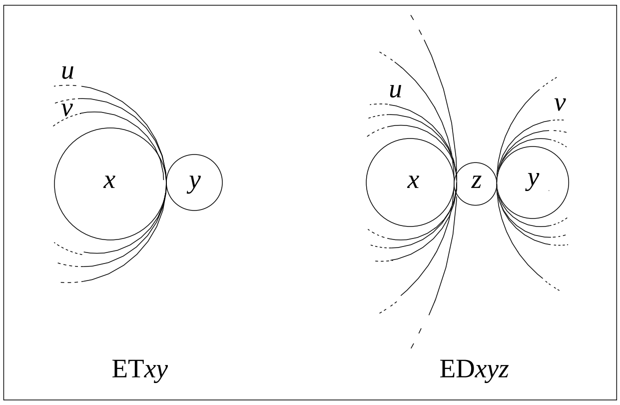

Figure 6. Externally tangent and externally diametrical spheres (D12, D14).

D15. $\mathrm{ID}xyz =_{\mathrm{df}} \mathrm{IT}xz \wedge \mathrm{IT}yz \wedge (\neg \mathrm{O}uz \wedge \neg \mathrm{O}vz \wedge \mathrm{ET}xu \wedge \mathrm{ET}yv \rightarrow \neg \mathrm{O}uv)$
(x and y are internally diametrical wrt z)

D16. $\mathrm{CNC}xy =_{\mathrm{df}} x = y \vee (\mathrm{PP}xy \wedge (\mathrm{ED}uvx \wedge \mathrm{IT}uy \wedge \mathrm{IT}vy \rightarrow \mathrm{ID}uvy)) \vee (\mathrm{PP}yx \wedge (\mathrm{ED}uvy \wedge \mathrm{IT}ux \wedge \mathrm{IT}vx \rightarrow \mathrm{ID}uvx))$
(x is concentric with y)

5.3 ALIGNMENT OF SPHERES

We can now easily exploit Tarski's definitions to define the notion of alignment among spheres, and to establish notions analogous to those of segments and triangles in terms of spheres (*s-segments* and *s-triangles*):

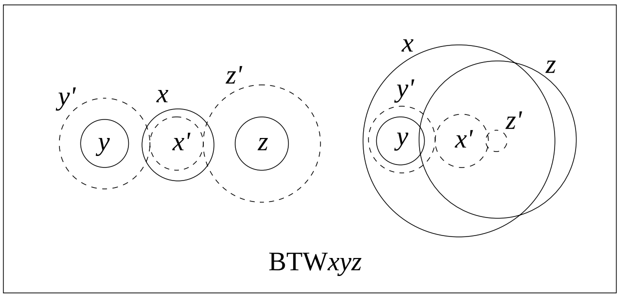

Figure 7. *x* is *between y* and *z* (D17).

D17. $\mathrm{BTW}xyz =_{\mathrm{df}} \exists x'y'z'(\mathrm{CNC}xx' \wedge \mathrm{CNC}yy' \wedge \mathrm{CNC}zz' \wedge \mathrm{ED}y'z'x')$
(x is between y and z)

D18. $\mathrm{LIN}xyz =_{\mathrm{df}} \mathrm{BTW}xyz \vee \mathrm{BTW}yxz \vee \mathrm{BTW}zxy$
(x is aligned wrt y and z)

D19. $\mathrm{SSD}xyz =_{\mathrm{df}} \mathrm{BTW}xyz \vee \mathrm{BTW}yxz \vee \mathrm{CNC}xy$
(x and y are on the same side wrt z)

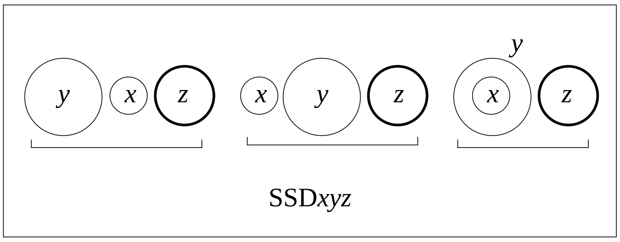

Figure 8. *x* is *on the same side* of *y* with respect to *z* (D19)

D20. $\mathrm{SEG}xy =_{\mathrm{df}} \neg \mathrm{CNC}xy$ *(x and y form an s-segment)*

D21. $\mathrm{TRI}xyz =_{\mathrm{df}} \neg \mathrm{CNC}xy \wedge \neg \mathrm{CNC}yz \wedge \neg \mathrm{CNC}xz \wedge \neg \mathrm{LIN}xyz$
(x, y and z form an s-triangle)

We restrict now the notions of s-segment and s-triangle in order to exclude the case where one of the spheres is part of some other one.

D22. $\mathrm{PNP2}xy =_{df} \neg \mathrm{P}xy \wedge \neg \mathrm{P}yx$

D23. $\mathrm{PNP3}xyz =_{df} \mathrm{PNP2}xy \wedge \mathrm{PNP2}xz \wedge \mathrm{PNP2}yz$

(pairwise not part)

D24. $\mathrm{PBTW}xyz =_{df} \mathrm{BTW}xyz \wedge \mathrm{PNP3}xyz$ *(proper between)*

D25. $\mathrm{PSEG}xy =_{df} \mathrm{SEG}xy \wedge \mathrm{PNP2}xy$ *(proper s-segment)*

D26. $\mathrm{PTRI}xyz =_{df} \mathrm{TRI}xyz \wedge \mathrm{PNP3}xyz$ *(proper s-triangle)*

5.4 RECONSTRUCTING STANDARD AXIOMS FOR CONGRUENCE

We can now introduce the proper axioms of congruence, modifying the formulation presented in (Coxeter 1989), which is a variation of Hilbert's system, by exploiting the parallelism between points and spheres.

A9. $\mathrm{CG}xy \wedge \mathrm{CG}zy \rightarrow \mathrm{CG}xz$

A10. $\mathrm{PSEG}xy \wedge \mathrm{SEG}zw \rightarrow \exists! x'y''(\mathrm{CG}(x+y)(x'+y') \wedge \mathrm{CNC}zx' \wedge \mathrm{CG}xx' \wedge \mathrm{SSD}wy'x')$

(Transportability of s-segments)

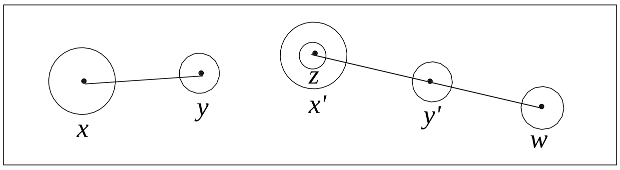

Figure 9. Transportability of s-segments (A10). Given a proper s-segment *xy* and an arbitrary segment *zw*, there exists a unique s-segment *x'y'* congruent to *xy* such that *x'* is concentrical to *z* and congruent to *x*, while *y'* is on the same side of *w* with respect to *x'*.

A11. $\mathrm{PBTW}yxz \wedge \mathrm{BTW}y'x'z' \wedge \mathrm{CG}xx' \wedge \mathrm{CG}(x+y)(x'+y') \wedge \mathrm{CG}(y+z)(y'+z') \rightarrow \mathrm{CG}(x+z)(x'+z')$

(Congruence of s-segments)

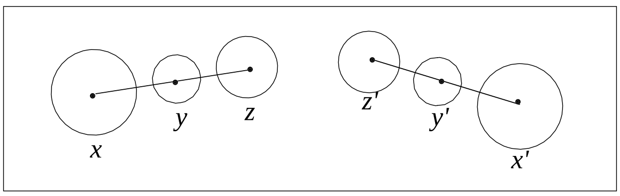

Figure 10. Congruence of s-segments (A11). If *xy* and *yz* are respectively congruent to *x'y'* and *y'z'*, then *xz* is congruent to *x'z'* provided that *xyz* and *x'y'z'* are aligned.

A12. $\mathrm{PTRI}xyz \wedge \mathrm{TRI}x'y'z' \wedge \mathrm{PBTW}yxv \wedge \mathrm{BTW}y'x'v' \wedge \mathrm{CG}xx' \wedge \mathrm{CG}yy' \wedge \mathrm{CG}zz' \wedge \mathrm{CG}vv' \wedge \mathrm{CG}(x+y)(x'+y') \wedge \mathrm{CG}(x+z)(x'+z') \wedge \mathrm{CG}(y+z)(y'+z') \wedge \mathrm{CG}(x+v)(x'+v') \rightarrow \mathrm{CG}(z+v)(z'+v')$

(Congruence in s-triangles)

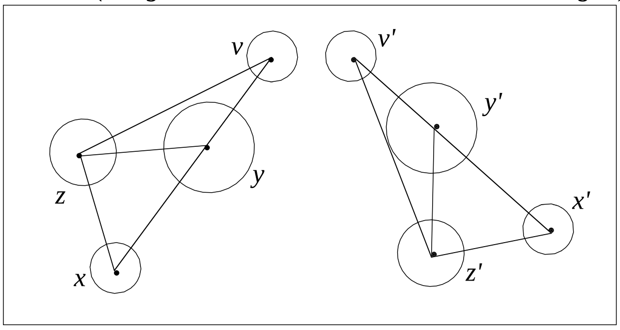

Figure 11. Congruence in s-triangles (A12). The s-segment *vz* is congruent to *v' z'* if the s-triangle *xyz* is congruent to the s-triangle *x'y'z'* and *xv* is congruent to *x'v'*.

The axiom A9, together with suitable existential assumptions, implies that 'CG' is an equivalence relation. Axioms A10-A12 are described by Figures 9-11.

The axioms above constrain the congruence between binary sums of spheres (s-segments).

5.5 CONGRUENCE BETWEEN ARBITRARY REGIONS

In the following, we first establish a notion of congruence between (suitable) sums of spheres, and then we use such a notion to constrain the congruence between arbitrary regions.

When considering the congruence between sums of spheres, we could follow the reasoning used by Hilbert in order to establish the congruence between sums of points (figures); to do that, we need to establish a suitable bijection between the two sums of spheres such that the corresponding s-segments are congruent. In order to avoid looking at all possible bijections (which would require a second-order axiomatization), we define an *ad hoc* congruence relation holding for *scalene* sums of spheres (sums of not s-connected spheres such that all of them are of different sizes). This means that, with respect to the size of the spheres, only one bijection exists. Once we have congruence between scalene sums of spheres, we use it to constrain congruence between arbitrary regions.

D27. $\Sigma SSx =_{df} \forall y(MCPyx \rightarrow (SPHy \wedge \neg\exists z\ (MCPzx \wedge \neg\ y{=}z \wedge CGzy)))$ *(scalene sum of spheres)*

D28. $\Sigma CGxy =_{df} \Sigma SSx \wedge \Sigma SSy \wedge$

$\forall z(MCPzx \rightarrow \exists w(MCPwy \wedge CGzw)) \wedge$
$\forall z(MCPzy \rightarrow \exists w(MCPwx \wedge CGzw)) \wedge$
$\forall uv(MCPux \wedge MCPvx \wedge SEGuv) \rightarrow$
$\exists u'v'(MCPu'y \wedge MCPv'y \wedge CG(u+v)(u'+v'))$
(congruent scalene sums of spheres)

A13. $CGxy \leftrightarrow$
$(\forall z(\Sigma SSz \wedge Pzx \rightarrow \exists w(\Sigma CGzw \wedge Pwy))$
$\wedge\ \forall z(\Sigma SSz \wedge Pzy \rightarrow \exists w(\Sigma CGzw \wedge Pwx)))$
(congruence of arbitrary regions)

Notice that the second and third lines in D28 constrain the two sums to have the same number of spheres. By means of A13 we can conclude that $\Sigma CGxy \rightarrow CGxy$.

5.6 FURTHER CONSTRAINTS

We need now to add some further axioms constraining the congruence between single spheres. Hilbert's axioms are of no help here, since he obviously assumes that all points are congruent to each other. Things are of course different in the case of spheres. A preliminary list of axioms is reported below.

A14. $CGxy \rightarrow \neg PPxy$

A15. $SPHx \wedge CGxy \rightarrow SPHy$

A16. $(CGxy \wedge IDxyz \wedge EDxyw) \rightarrow CNCzw$

A14 excludes the congruence of two spheres being one internal to the other; A15 states that congruence preserves the property of being a sphere, while A16 allows us to establish the concentricity of two spheres (Figure 12).

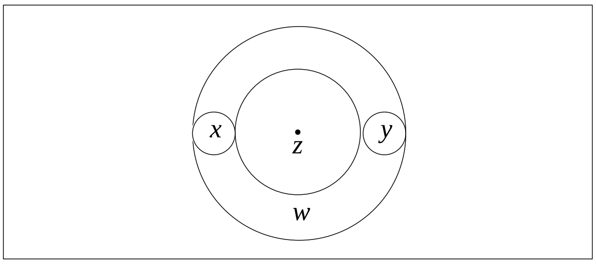

Figure 12. if x and y are congruent then *z* and *w* are concentric spheres (A16)

We must now add some *existential axioms*, which are necessary to ensure that enough spheres exist:

A17. $(SEGxy \wedge \neg Oxy \wedge \neg ETxy) \rightarrow \exists z\ (EDxyz)$

A18. $ETxy \rightarrow \exists z\ (IDxyz)$

A19. $SPHx \rightarrow \exists yz\ (IDyzx \wedge ETyz \wedge CGyz)$

A17 guarantees that, given two not connected spheres, a third sphere always exists between them; A18 states that the minimum sphere including two tangent spheres always exists. A19 states that two congruent tangent spheres, internally diametrical to a given one, always exist. Finally, two last axioms guarantee that our space is tridimensional, by stating that four mutually tangent spheres exist, while five mutually tangent spheres do not exist.

A20.

$\exists xyzw\,(\mathrm{CG}xy \wedge \mathrm{CG}xz \wedge \mathrm{CG}xw \wedge \mathrm{ET}xy \wedge \mathrm{ET}xz \wedge \mathrm{ET}xw \wedge \mathrm{ET}yz \wedge \mathrm{ET}yw \wedge \mathrm{ET}zw)$

A21.

$\forall xyzwv\, \neg\,(\mathrm{CG}xy \wedge \mathrm{CG}xz \wedge \mathrm{CG}xw \wedge \mathrm{CG}xv \wedge \mathrm{ET}xy \wedge \mathrm{ET}xz \wedge \mathrm{ET}xw \wedge \mathrm{ET}xv \wedge \mathrm{ET}yz \wedge \mathrm{ET}yw \wedge \mathrm{ET}yv \wedge \mathrm{ET}zw \wedge \mathrm{ET}zv \wedge \mathrm{ET}wv)$

5.7 USING THE CONGRUENCE PRIMITIVE

We are now in the position to define l- and p-connection (Figure 13), and then the usual notion of connection:

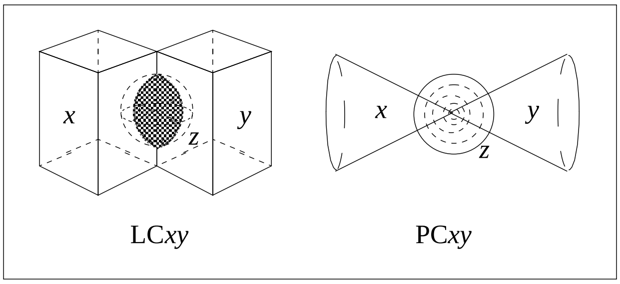

Figure 13. Line– and point-connection.

D29 $\mathrm{LC}xy =_{\mathrm{df}} \neg\mathrm{SC}xy \wedge \exists z(\mathrm{SPH}z \wedge \mathrm{O}zx \wedge \mathrm{O}zy \wedge \mathrm{SR}(z\text{-}x) \wedge \mathrm{SR}(z\text{-}y) \wedge \neg\mathrm{SR}(z\text{-}(x+y)))$ *(l-connection)*

D30. $\mathrm{PC}xy =_{\mathrm{df}} \neg\mathrm{SC}xy \wedge \neg\mathrm{LC}xy \wedge \exists z(\mathrm{SPH}z \wedge \forall u(\mathrm{CNC}uz \rightarrow (\mathrm{O}ux \wedge \mathrm{O}uy)))$ *(p-connection)*

D31. $\mathrm{C}xy =_{\mathrm{df}} \mathrm{SC}xy \vee \mathrm{LC}xy \vee \mathrm{PC}xy$ *(connection)*

Congruence allows us to easily define the notion of *convexity* for region (Figure 14):

D32. $\mathrm{CONV}x =_{\mathrm{df}} \forall uvw(\mathrm{P}(u+v)x \wedge \mathrm{CG}uv \wedge \mathrm{CG}wu \wedge \mathrm{BTW}wuv) \rightarrow \mathrm{P}wx$ *(x is convex)*

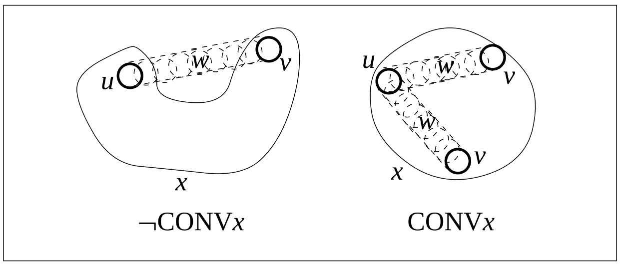

Figure 14. Defining convexity.

Here there are some theorems stating interesting properties of some morphological relations. T18 shows that there exists only one sphere concentric with y and congruent to x. T19 states that 'ET' relation is preserved by congruence. T20-T23 establish fondamental properties of the 'BTW' relation. T24 shows that spheres are convex regions.

T17. $\mathrm{CNC}xy \rightarrow \mathrm{P}xy \vee \mathrm{P}yx$ (D16)

T18. $(\mathrm{SPH}x \wedge \mathrm{SPH}y) \rightarrow \exists! z(\mathrm{CG}zx \wedge \mathrm{CNC}zy)$ (A10; T17)

T19. $(\mathrm{SPH}x \wedge \mathrm{ET}xy \wedge \mathrm{CG}(x+y)(x+z)) \rightarrow \mathrm{ET}xz$ (A10; A11)

T20. $\mathrm{BTW}xyz \leftrightarrow \mathrm{BTW}xzy$ (D17)

T21. $\exists yz(\mathrm{BTW}xyz)$ (A10; A17; A20; T19)

T22. $\mathrm{BTW}xyz \rightarrow (\neg\mathrm{BTW}yxz \wedge \neg\mathrm{BTW}zyx)$ (A18; A19)

T23. $(\mathrm{BTW}yxz \wedge \mathrm{BTW}zyu) \rightarrow (\mathrm{BTW}yxu \wedge \mathrm{BTW}zxu)$ (A10; A16; A17; A18)

T24. $\mathrm{SPH}x \rightarrow \mathrm{CONV}x$ (A10; A11; D32)

6 DISCUSSION

We have presented a logical theory of space quite rich in axioms and definitions. As stated in the introduction, its main purpose is to characterize the intended meaning of the three primitives used – parthood, simple region, and congruence – in a domain where only three-dimensional regular regions are assumed. However, even if the formal properties of this theory (i.e. its soundness, its completeness and its computational properties) have yet to be studied in detail, the important theorems coming out of the proposed axiomatization let us describe some fundamental features of space.

In this preliminary work, we have concentrated on the task of explaining in detail our ontological assumptions about space by means of formal axioms, exploring at the same time the expressive power, the mutual relationships and the *cognitive relevance* of the primitives adopted.

Let us now add some comments on the comparison between our theory and the RCC theory. We observe first that the latter is a *minimal* theory of connection: this choice may have the advantage of some computational properties, but it is less precise in the meaning of the primitives assumed. Our theory, on the other hand, characterizes more precisely the primitives. In particular, RCC's 'C' primitive can be interpreted as denoting strong-connection, line-connection, point-connection or a combination of these, while we have shown that the above axiomatization prevents the interpretation of our 'SC' as line-connection or point-connection.

This means that the definitions of point-connection, "doughnut" and "quasi-manifold" discussed in (Gotts 1994, Gotts 1994, Cohn 1995) only hold in the *intended* model where two regions are connected if they share at least one point; but if we consider a model were 'C' is interpreted as s-connection then the definitions do not capture the desired meanings. This freedom in the interpretation could be an advantage for the RCC approach, in the sense that the theory is apt to capture a very general notion of connection, which may be useful for various purposes. However, the theory appears to be too weak for a formal characterization of space in its present state.

Full Clarke's theory, as recently shown in (Asher and Vieu 1995), is surely more satisfactory in this respect, but it pays the price of committing to the classical distinction between open and closed regions, which many people consider debatable from the cognitive point of view. Moreover, as discussed in (Varzi 1996b), full Clarke's theory presents some unpleasant mereological properties, since "an open region is always a proper part of its own closure, but there is no mereological difference between the two".

In order to find a satisfatory solution to these problems, our strategy has been to emphasize the role played by the *morphological* properties of space. The definition of sphere (D11) appears to us to be emblematic in this respect. Of course, it requires a characterization of congruence, and this is a complicated task. We admit that the methodology adopted to have an axiomatization of congruence turns out to be complicated and difficult, but unfortunately the alternative would have been "inventing" our axioms for congruence between regions merely by means of introspection. In the literature we are aware of, the only proposal in this sense is the axiomatization of "convex hull" used within the RCC theory, but the same authors admit it cannot be considered as satisfactory yet. The approach we have developed offers a characterization of a *very* powerful primitive and seems to us amenable both for concrete applications and for further mathematical speculations.

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