Quantificational modal logic with sequential Kripke semantics

(prefinal version)

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ABSTRACT. *We introduce quantificational modal operators as dynamic modalities with (extensions of) Henkin quantifiers as indices. The adoption of matrices of indices (with action identifiers, variables and/or quantified variables as entries) gives an expressive formalism which is here motivated with examples from the area of multi-agent systems. We study the formal properties of the resulting logic which, formally speaking, does not satisfy the normality condition. However, the logic admits a semantics in terms of (an extension of) Kripke structures. As a consequence, standard techniques for normal modal logic become available. We apply these to prove completeness and decidability, and to extend some standard frame results to this logic.*

KEYWORDS: *quantificational modality, dynamic logic, Henkin quantifiers, modal logic, multi-agent systems.*

1. The modal approach in multi-agent systems

In the last 20 years, a variety of logical systems have been developed for modeling agents. Building on the pioneering work of Hintikka [HIN 64], most of them address the description of situations where there is just one agent with peculiar attitudes, knowledge, believes, and abilities. The same logical formalism is sometimes adopted for modeling systems that comprise more than one agent, called *multi-agent systems* (MAS). However, the complexity of the cases in which several agents act concurrently, perhaps affecting each other, cannot be reduced to the description of each agent in isolation. Indeed, in addressing the description of a community, researchers have been exploiting different formalisms in order to pinpoint the specificity of the agents and their actions as well as the interactions among them and their strategies in evolving environments.

Starting with Hintikka [HIN 64] and later with the works of Moore [MOO 85], Cohen and Levesque [COH 90], Rao and Georgeff [RAO 91], modal logic has become the formal language for studying agents. Still, as of today [HOE 03, HOE 02], the work done in the (modal) description of communities of agents seem scattered in a variety of different logics and, we surmise, this might be due to the inadequacy of standard logic tools. Indeed, researchers have been testing the modal formalism in a variety of ways [D'I 97, BEN 02] by studying the interactions among several modal operators, by varying the semantic interpretations, by isolating techniques for combining modal systems (like fusion, merging, embedding, full-fibring) [GAB 03]. However, the systems developed for truly concurrent multi-agent systems seem of limited application.

One can explain this situation in different ways. We report here two reasons that, in our view, are crucial. On the one hand, the combination of modal languages often gives rise to complicated logics which, unless severely restricted, are hard to study and do not present nice formal properties [BEN 02]. In turn, this affects the adoption and the applicability of the formalisms; a serious obstacle to what we could dub the *tiling approach in* MAS, that is, the attempt of describing complex agents by assembling different modal operators, one operator for each (independent) feature of the agents. On the other hand, formal systems that contain different modalities are hard to compare both at the syntactic and at the semantic levels. In this way, it is hard to state the advantages of one approach over the others in a general perspective. Because of this situation, today we lack a systematic methodology for comparison of multi-agent systems. In truth, sometimes one can overcome the problem by embedding several multi-agent logics into the same (stronger) language whose formal properties are known. An example is given by the results in [SCH 98]. However, this approach is hardly generalizable.

Toward flexible languages

This being the state of the research, new approaches should be studied to move forward in this area and, in doing this, it might be important to take advantage of the semantic flexibility of modal logic. Indeed, we consider an advantage of modal logic the fact that modal operators support a variety of interpretations. This flexibility is sometimes exploited to describe different MAS systems in the same language by varying the semantics adopted. The result is a uniform syntactic description of a variety of systems avoiding the application of ad hoc formalisms which are often hard to relate. Through this common language, a comparative study of the modeled systems becomes possible by concentrating on the analysis of their semantics.

Starting from this overall view, we look for new modal operators that are rich in expressive power, easily related to different semantics, and suitable for modeling MAS. We propose modalities that are obtained by integrating two different elements, namely the modal operators of dynamic logic [HAR 00] and the Henkin quantifiers [HEN 61, KRY 95]. The operators we study are modalities that describe the evolution of the MAS in the fashion of dynamic logic, and that contain (free and quantified) variables. Informally, the role of quantifiers is to mark the attitude of an agent at a certain point in time while constants register actions the agents must execute (they have no choice to make) at that point in time. In this paper, we present the characteristics of the new operators, define the corresponding language, and describe one semantic interpretation. This interpretation seems particularly interesting for its logical properties and is suitable for describing cooperative agents and coalitions [OSB 94]. The comparison of this logic with other modal systems for MAS (an interesting relationship exists with the logic in [PAU 02]) as well as other semantic interpretations (see [BOR 03] for an example) are not discussed here.

The paper is structured as follows. In section 2 we look at dynamic logic (DL) as a formalism for agents and justify the introduction of complex (constant) operators to deal with MAS. This change gives a new perspective on DL. For this reason, a throughout comparison of propositional DL and the new system is provided. Section 3 takes a step further and introduces the quantificational modal operators. A semantic interpretation suitable for cooperative agents is adopted; formal and application-driven examples of the new formalism are given. Section 4 presents the axiomatization followed by completeness and decidability results. Toward the end of the section, we discuss the relationship between our quantificational logic and propositional normal logics. Finally, we show how other results for normal logics are inherited in our system by considering some standard frame properties.

2. From *PDL* to *multi-agent PDL*

2.1. *Carving up* PDL

Our system can be introduced as the generalization of a propositional multi-modal logic in the sense of [GAB 03], that is, a subsystem of *Propositional Dynamic Logic*, (*PDL*) [HAR 00] that is here dubbed *restricted PDL* (*rPDL*).

PDL is a logical system developed to describe properties of interaction between programs and propositions independently from the domain of computation. The system blends modal logic and the algebra of regular expressions into a formalism that has found broad application also outside the field of logics of programs. Here is a short introduction to the fragment *rPDL* relevant to our work (and some standard definitions to be specialized later to our system). For a complete introduction of *PDL* and more traditional extensions see [HAR 00].

2.1.1. *The fragment* rPDL

The language of *rPDL* contains a non-empty countable set of *proposition identifiers*, *PropId*, and a non-empty countable set of *action identifiers*, *ActId*. We shall always assume that these sets are disjoint.

The proposition identifiers are the simplest (atomic) formulas in the language. Complex formulas are built recursively as in propositional logic (using implication $\rightarrow$, and negation $\neg$) and via modal operators as described below. We shall make use of the standard conventions for $\wedge, \vee, \leftrightarrow, \top$ (the *truth*), and $\perp$ (the *falsum*). In *PDL*, combinations of action identifiers are obtained recursively by applying several constructs. In *rPDL*, we limit the language to the *composition construct* (;) only.[1] The composition of a and b is indicated by $a; b$ or simply by $a\, b$. Finally, modality is introduced through the necessity operators $[a]$ where a is an action identifier or a composition of action identifiers. Since action identifiers and their compositions are used to identify modalities in the logic, they play the crucial role of *modality identifier*.

The modality identifiers for *rPDL* are defined recursively:

1) all elements in *ActId* are modality identifiers

2) $a; b$ (equivalently, $a\, b$) is a modality identifier if a and b are both modality identifiers

The set of *formulas* is the smallest set satisfying the following clauses:

1) all elements in *PropId* are formulas (atomic formulas)

2) $\neg\varphi$ and $\varphi \rightarrow \psi$ are formulas if φ and ψ are formulas

3) $[a]\varphi$ is a formula if a is a modality identifier and φ is a formula

In case 2), we say that φ is in *antecedent position* (with respect to ψ) and that ψ is in *consequent position* (with respect to φ). Also, we adopt the standard notion of *subformula*. These latter notions will apply to all the languages introduced in this paper without further comments.

2.1.2. *The semantics of* rPDL

The semantics of *rPDL* is taken from *PDL* and, more generally, from Kripke's semantics for modal logic. We depart slightly from the usual presentation in as much as this allows us to present a notion of frame suitable to our tasks in later sections.

Recall that a *Kripke Frame* is a pair $\langle W; R\rangle$ where W is a set of elements called *states* and R a function assigning a binary relation on W to each modal operator in the language. The following definition of *Agent Kripke Frame* extends this notion.

DEFINITION 1 (AGENT KRIPKE FRAME). —

An Agent Kripke Frame *for rPDL is a triple* $\mathcal{K} = \langle W, Act; R\rangle$ *where:*

1) W *is a non-empty set* (the set of states)*,*

2) Act is a non-empty set (the set of actions)*, and*

1. Thus, here we do not consider most constructs of *PDL* like "choice" ($\cup$), "iteration" or "star" ($*$), and "test" (?).

3) for all $\alpha \in Act$, $R(\alpha)$ *is a binary relation on* W (the accessibility relation for action α).

We use $\alpha, \beta, \ldots$, possibly decorated, for the elements in *Act*.

Historically, *PDL* interprets action identifiers, and more generally all modality identifiers, as *programs*. Since we are interested in agents and their behavior, we will interpret action identifiers more broadly as *actions*. Informally, the set *Act* plays the role of the set of actions that the agents can perform. Note that the action identifiers are rigid designators.

Formulas of *rPDL* are interpreted over Agent Kripke frames. The truth-value of a formula depends on the chosen *valuation function* $[\![\cdot]\!]$ which intervenes to assign a subset of W to each atomic proposition and an action in *Act* to each action identifiers. Since the set of atomic propositions and the set of actions identifiers are disjoint, we can safely use the same notation for the valuation function over the two sets.

An agent Kripke frame augmented with a valuation function is called a *structure*.

DEFINITION 2 (AGENT KRIPKE STRUCTURE FOR *rPDL*). —

An agent Kripke structure for *rPDL is a 4-tuple* $\mathcal{M} = \langle W, Act; R, [\![\cdot]\!]\rangle$ *where:*

1) $\langle W, Act; R\rangle$ *is an agent Kripke frame and*

2) $[\![\cdot]\!]$ *is a function* (the valuation function) *such that* $[\![p]\!] \subseteq W$ *for* $p \in PropId$ *and* $[\![a]\!] \in Act$ *for* $a \in ActId$.

Given an agent Kripke frame $\mathcal{M}$, we extend $[\![\cdot]\!]$ to all modality identifiers and R to sequences of actions as follows:[2]

$$[\![ab]\!] =_{def} [\![a]\!][\![b]\!]$$
$$R(\alpha\beta) =_{def} R(\alpha) \circ R(\beta)$$

Let $\mathcal{M}$ be a structure. We write $\mathcal{M}, s \models_{rPDL} \varphi$ to say that φ is true at state s of $\mathcal{M}$ (and $\mathcal{M}, s \not\models_{rPDL} \varphi$ if φ is false). The semantic relation $\models_{rPDL}$ is defined as follows:

a) Let $p \in PropId$, then $\mathcal{M}, s \models_{rPDL} p$ if $s \in [\![p]\!]$

b) $\mathcal{M}, s \models_{rPDL} \neg\varphi$ if $\mathcal{M}, s \not\models_{rPDL} \varphi$

c) $\mathcal{M}, s \models_{rPDL} \varphi \rightarrow \psi$ if $\mathcal{M}, s \not\models_{rPDL} \varphi$ or $\mathcal{M}, s \models_{rPDL} \psi$

d) $\mathcal{M}, s \models_{rPDL} [a]\varphi$ if for all $t \in W$ such that $(s, t) \in R([\![a]\!])$, $\mathcal{M}, t \models_{rPDL} \varphi$

We write $\mathcal{M} \models_{rPDL} \varphi$ to say that formula φ is *valid in* $\mathcal{M}$, that is, it is true at each state of structure $\mathcal{M}$.

DEFINITION 3 (AGENT KRIPKE MODEL IN *rPDL*). —

2. Operator $\circ$ is the usual relational composition and it is associative. On binary relations it is defined by $R \circ S = \{(u, v) \mid \exists w(u, w) \in R$ and $(w, v) \in S\}$.

An agent Kripke model *for a set of formulas* Σ *in* rPDL *is a structure* $\mathcal{M}$ *for* rPDL *such that all formulas* $\varphi \in \Sigma$ *are valid in* $\mathcal{M}$.

2.1.3. *The logic of* rPDL

Since we are dealing with a fragment of *PDL*, we provide a formalization of the restricted language only.[3] In practice, we disregard the axiom schemas about the other operators on action identifiers. Furthermore, we use all standard logic connectives since these are definable from $\to$ and $\neg$.

As usual, a formula φ *is provable* in a set of formulas Σ (is a *theorem* of Σ) if $\varphi \in \Sigma$. If φ is provable in Σ, we write $\vdash_\Sigma \varphi$. We write $\vdash \varphi$ when Σ is clear from the context.

(1) Axioms for propositional logic

(2) $[a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$ (Normality)

(3) $[ab]\varphi \leftrightarrow [a][b]\varphi$ (Composition)

(MP) $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$ (Modus Ponens)

(Nec) $\dfrac{\vdash \varphi}{\vdash [a]\varphi}$ (Necessitation)

The *logic of rPDL* is the smallest set of formulas in *rPDL* containing all instances of $(1) - (3)$ and closed under rules (MP) and (Nec). We write Λ_r for the logic of *rPDL*.

DEFINITION 4 (FRAME-SOUNDNESS). —

Let $\mathcal{F}$ *be a class of frames. A logic* Λ *is* sound with respect to $\mathcal{F}$ *if*

1) all formulas φ *in* Λ *are valid for structures with frame in* $\mathcal{F}$*, and*

2) all the rules are truth-preserving.

In other terms, each structure with frame in $\mathcal{F}$ *is a model for* Λ.

The proof that the logic Λ_r is sound with respect to the class of agent Kripke frames is routine [CHE 80].

DEFINITION 5 (FRAME-COMPLETENESS). —

Let $\mathcal{F}$ *be a class of frames for a language* $\mathcal{L}$*. A logic* Λ *is* complete for $\mathcal{L}$ with respect to $\mathcal{F}$ *if the formulas of* $\mathcal{L}$ *valid in the structures with frames in* $\mathcal{F}$ *are provable in the logic.*

DEFINITION 6 (FRAME-DECIDABILITY). —

3. The first satisfactory axiomatization of full *PDL* was provided by Segerberg [SEG 77]. Completeness proofs are given in [GAB 77, PAR 78], see also [HAR 00].

Let $\mathcal{F}$ be a class of frames. A logic Λ is decidable with respect to $\mathcal{F}$ *if there exists an algorithm that, given a formula φ, determines whether there is a state of a model for Λ (with frame from the class) at which φ is true.*

It turns out that the logic of *rPDL* is complete and decidable with respect to the class of all agent Kripke frames.

A simple strategy to prove completeness (and decidability as well) follows from the observation that *rPDL* is the *fusion* (see [GAB 03]) of countably many *normal logics*[4]. Note that in this sublanguage of *PDL*, axiom (3) has no much deductive import; it simply tells us that modal operators with complex identifiers are nothing more than sequences of simple modalities. Fusion preserves the *finite model property*[5] as well as completeness and decidability [GAB 03], and this allows us to conclude that *rPDL* is complete and decidable (with respect to the class of agent Kripke frames) provided we verify a couple of properties:

(a) Any agent Kripke frame, with $n \in \mathbb{N}^+ \cup \{+\infty\}$ distinct action identifiers, is the fusion of n standard Kripke frames (and vice versa),

(b) *rPDL* has the finite model property.

The latter follows by applying the standard argument (adapted from mono-modal logics) to formulas of *rPDL*, while the first property follows from the definitions.

A detailed discussion of the finite model property, completeness, and decidability in mono- and multi-modal logics, including a discussion of the *PDL* system, can be found in [BLA 01, CHE 80].

PROPOSITION 7. — *The logic of rPDL is sound, complete, and decidable with respect to the class of agent Kripke frames.*

2.2. *Making room for agents:* mPDL

We modify *rPDL* in order to describe multi-agent systems. The resulting language differs from *rPDL* in the set of modality identifiers. We dub this system *multi-agent Propositional Dynamic Logic* (*mPDL*).

We want the changes in the state of a multi-agent system to be the result of the actions performed by the agents in that system. To preserve (true) concurrency, our logic should be able to make a distinction between actions performed at the same point in time and actions performed at different times. Also, since agents may have different capabilities and responsibilities, it should be possible to have different outputs when

4. A modal logic, built upon classical propositional logic, is *normal* if the modalities in it satisfy the normality axiom (2) and the necessitation rule (Nec).

5. A modal logic has the finite model property (with respect to a class of models) if for each formula φ true in some model of the class there is a finite model $\mathcal{M}$ in the same class such that φ is true at some state in $\mathcal{M}$.

the very same action is performed by different agents. This last desideratum brings the need for an explicit tie between actions and their performers.

From the above general perspective, the simple action identifier adopted by *rPDL* seems unable to play the role of a modality identifier. Our logic looks at modality identifiers as ordered sets of actions identifiers assuming that, in order to understand how the system evolves, one must know which action identifier is associated to which agent at that point in time.

Assume the set of proposition identifiers, *PropId*, and the set of action identifiers, *ActId*, have been fixed as in the previous section. We fix a positive integer k which informally is the number of agents in the multi-agent system we want to describe. The guiding idea is that *rPDL* is the formal system corresponding to $k = 1$.

Choose an ordering of the k agents in the multi-agent system. We write **1** for the first agent, ..., **k** for the k-th agent. If agent **1** performs the action denoted by a_1, agent **2** the action denoted by $a_2, \ldots,$ agent **k** the action denoted by a_k, write

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

for the modal operator that captures the evolution of the system according to the concurrent execution of actions $[\![a_1]\!], \ldots, [\![a_k]\!]$ by agents $\mathbf{1}, \ldots, \mathbf{k}$, respectively.

Assuming $a_1 \neq a_2$, operator $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$ and operator $\begin{bmatrix} a_2 \\ a_1 \\ \vdots \\ a_k \end{bmatrix}$ (differing only in the first two entries) are different modality identifiers since action identifiers a_1 and a_2 are associated with different agents. In short, we not only list all the (concurrent) action identifiers, but also link each action identifier to the agent performing the corresponding action. Since one can describe successive actions executed by the agents by using multi-columns modal operators, we are led to the following

DEFINITION 8 (MODALITY IDENTIFIER FOR *mPDL*). —

A modality identifier for *mPDL is a $k \times n$-matrix ($n \geq 1$)*

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{matrix}$$

where a_{ij} is an action identifier (a_{ij} and a_{rs} not necessarily distinct).

The set of formulas of *mPDL* is the smallest set satisfying the following clauses:

(1) all elements in *PropId* are formulas (atomic formulas)

(2) $\neg\varphi$ and $\varphi \rightarrow \psi$ are formulas if both φ and ψ are formulas

(3) $[A]\varphi$ is a formula if A is a modality identifier for *mPDL* and φ is a formula

The semantics and the axiomatization of *mPDL* will follow naturally from those of *rPDL* once we state the semantic counterpart of a modality identifier for *mPDL*. In *rPDL*, the interpretation of a modality identifier is a (perhaps complex) action, and so a transition in the structure. Similarly, the interpretation of a modality identifier in *mPDL* should be associated with a transition in the structure for *mPDL*. This is the rationale for the following definitions.

DEFINITION 9 (k-ACTION). —

Given a set Act of actions, a basic k-action for *mPDL is any column of* k *elements in Act*

$$\begin{matrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{matrix}$$

k-Actions *are given recursively by:*

1) all basic k-actions are k-actions

2) AB is a k-action if A and B are both k-actions

DEFINITION 10 (MULTI-AGENT KRIPKE FRAME FOR *mPDL*). —

A multi-agent Kripke frame for *mPDL is a triple* $\mathcal{K} = \langle W, Act; R\rangle$ *where:*

1) W is a non-empty set (the set of states),

2) Act is a non-empty set (the set of actions), *and*

3) R is a function, (the accessibility relation), *mapping basic k-actions (over Act) to binary relations on W*

$$R\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} \subseteq W \times W.$$

DEFINITION 11 (MULTI-AGENT KRIPKE STRUCTURE FOR *mPDL*). —

A multi-agent Kripke structure for *mPDL is a 4-tuple* $\mathcal{M} = \langle W, Act; R, [\![\cdot]\!]\rangle$ *where:*

1) $\langle W, Act; R\rangle$ is a multi-agent Kripke frame for mPDL and

2) $[\![\cdot]\!]$ is a function (the valuation function) *such that* $[\![p]\!] \subseteq W$ *for* $p \in PropId$ *and* $[\![a]\!] \in Act$ *for* $a \in ActId$.

Given a multi-agent Kripke structure $\mathcal{M}$, $[\![\cdot]\!]$ is extended inductively to all modality identifiers in the language *mPDL* by

$$\left[\!\!\left[\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{matrix}\right]\!\!\right] =_{def} \begin{matrix} [\![a_{11}]\!] & [\![a_{12}]\!] & \ldots & [\![a_{1n}]\!] \\ [\![a_{21}]\!] & [\![a_{22}]\!] & \ldots & [\![a_{2n}]\!] \\ \vdots & \vdots & & \vdots \\ [\![a_{k1}]\!] & [\![a_{k2}]\!] & \ldots & [\![a_{kn}]\!] \end{matrix}$$

In what follows, we usully write $[\![A]\!]$ also for $[\![[A]]\!]$.

The truth-value of a formula is defined recursively by the following clauses:

a) Let $p \in PropId$, then $\mathcal{M}, s \models p$ if $s \in [\![p]\!]$

b) $\mathcal{M}, s \models \neg\varphi$ if $\mathcal{M}, s \not\models \varphi$

c) $\mathcal{M}, s \models \varphi \to \psi$ if $\mathcal{M}, s \not\models \varphi$ or $\mathcal{M}, s \models \psi$

d) $\mathcal{M}, s \models [A]\varphi$ if for all $t \in W$ such that $(s, t) \in R([\![A]\!])$, $\mathcal{M}, t \models \varphi$

DEFINITION 12 (MULTI-AGENT KRIPKE MODEL IN *mPDL*). —

A multi-agent Kripke model *for a set of formulas* Σ *in* mPDL *is a structure* $\mathcal{M}$ *for* mPDL *such that all formulas* $\varphi \in \Sigma$ *are valid in* $\mathcal{M}$.

The *logic of mPDL* is the smallest set containing the following axiom schemas $(1) - (3)$ and closed under the rules (MP) and (Nec). Here A and B stand for modality identifiers:

(1) Axioms for propositional logic

(2) $[A](\varphi \to \psi) \to ([A]\varphi \to [A]\psi)$ (Normality)

(3) $[AB]\varphi \leftrightarrow [A][B]\varphi$ (Composition)

(MP) $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$ (Modus Ponens)

(Nec) $\dfrac{\vdash \varphi}{\vdash [A]\varphi}$ (Necessitation)

The remaining notions (e.g. valid formula, theorem, etc.) are analogous to those for *mPDL*, see sections 2.1.2 and 2.1.3.

To highlight the correspondence with the semantics and axiomatization of *rPDL*, one can write $\overline{a_1 a_2 \ldots a_k}$ to denote the modality identifier $\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{matrix}$, $\overline{a_2 a_1 \ldots a_k}$ for the modality identifier $\begin{matrix} a_2 \\ a_1 \\ \vdots \\ a_k \end{matrix}$, and so on.

In this way, using set $\{\ \overline{a_1 a_2 a_3 \ldots a_k}\,,\ \overline{a_2 a_1 a_3 \ldots a_k}\,,\ \ldots\}$ for *ActId*, we obtain a *rPDL* language as described in section 2.1 which is equivalent to the *mPDL* language we started with. One sees that the semantic clauses and the logic of *rPDL* match semantics and deduction system of our *mPDL*. Soundness, completeness, and decidability of *mPDL* follow from the corresponding properties of *rPDL*.

PROPOSITION 13. — *The logic of mPDL is sound, complete, and decidable with respect to the class of multi-agent Kripke frames.*

This observation concludes the introduction of our multi-agent propositional dynamic logic which is, as we have seen, a propositional dynamic logic for true concurrency.

At this point one wonders why we needed to spend time on introducing a language with such a complex type of modal operators. After all, we just showed that the system obtained by using matrices of action identifiers as modality identifiers is nothing more than a fragment of *PDL* in the sense of section 2.1. The advantage of this approach will become evident when we introduce quantifiers in section 3. For the time being, it suffices to know that, although technically we have no advantage in using modality identifiers versus simple action identifiers, the change in perspective introduced by the explicit reference to agents and their actions is at the core of the quantificational language we are going to study.

2.3. rPDL *vs* mPDL

In the framework of propositional dynamic logic, there are two main strategies to represent concurrency.

In one case, the syntax remains that of *PDL* while the semantics is enriched (with a corresponding extension of the axiomatization). The strategy consists in imposing concurrency by isolating a subclass of the standard frames over which one should interpret the language. New axioms are added to match this restrictions. They basically state that (some or all) actions can be executed in any order without changing the set of reachable states. These are usually called *confluence axioms* and furnish a notion of concurrency. For instance, over action identifiers a and b, the axiom can be stated as follows:

$$(Conc^1_{ab}) \qquad \langle a\rangle[\,b\,]\,\varphi \;\rightarrow\; [\,b\,]\langle a\rangle\varphi$$

It should be clear that this strategy is not specific to dynamic logic. Indeed, it can be applied to poly-modal logics in general. However, it is important to notice that this characterization of concurrency is quite weak.

Dynamic logic provides a simple way of capturing a stronger notion of concurrency. By composing action identifiers, one can express concurrency as follows:[6]

$$(Conc^2_{ab}) \qquad [\, a\, b\,]\, \varphi \;\leftrightarrow\; [\, b\, a\,]\, \varphi$$

This formula matches the following constraint on frames:

$$R(\llbracket a \rrbracket) \circ R(\llbracket b \rrbracket) = R(\llbracket b \rrbracket) \circ R(\llbracket a \rrbracket)$$

Such a technique suffices, for instance, in describing independent assignments in a programming language. Let $\llbracket a \rrbracket$ be *assign value 1 to variable x* and $\llbracket b \rrbracket$ be *assign value 2 to variable y*. Formula $(Conc^2_{ab})$ "forces" concurrency by stating that the order of execution of a and b does not matter.

However, these conditions are useless when the actions involved are not independent, a quite natural condition in MAS. For instance, let action identifier $\llbracket a \rrbracket$ be *hold the nail against the wall* and let $\llbracket b \rrbracket$ be *hit the nail with the hammer*. Then, no matter which action we choose to execute first, the result is anyway different from the *actual* concurrent execution of the very same actions.

The other strategy to handle concurrency in *PDL* consists in enriching the language with new constructs (on action identifiers) that characterize concurrency explicitly. A couple of constructs have been proposed at about the time Dynamic Logic was developed. These are known as the *intersection* $(\cap)$ [HAR 00] and the *concurrency connective* $(\wedge)$[7] [PEL 87b, GOL 89] constructs.

Intuitively, for any two action identifiers a and b, the modality $[\, a \cap b\,]$ takes the system to those states that both the actions $\llbracket a \rrbracket$ and $\llbracket b \rrbracket$ admit. This interpretation fits quite well with the semantics of *PDL* but it is too restrictive for our tasks. In a sense, it requires $\llbracket a \rrbracket$ and $\llbracket b \rrbracket$ to be compatible actions. Indeed, this operator cannot deal with two actions $\llbracket a \rrbracket$ and $\llbracket b \rrbracket$ such that the first, when executed without interference, leads to states that are incompatible with the execution of the latter. The action of *pouring milk* into my empty cup and the action of *pouring coffee* into my empty cup do not lead to an inconsistent state when performed concurrently. However, if one executes the first only the system ends up in states that are inconsistent with the states reachable through the second action only. This very fact makes useless the application of $\cap$ to these actions.

The construct $\wedge$ on action identifiers was introduced by Peleg shortly after dynamic logic was recognized as a mature formalism and requires major changes in the semantics of *PDL*.[8] Following the work of Peleg, the semantics of the logic en-

6. In some cases the "shuffle" construct ($||$) is introduced to capture this type of concurrency. Modality $[\, a||b\,]$ corresponds to modality $[\, ab\,]$ further constrained by (Conc$^2_{ab}$).
7. Actually, Peleg used symbol $\cap$ as well. We change it to $\wedge$ for clarity. As a consequence, here the symbol $\wedge$ is overloaded. It corresponds to Peleg's concurrent construct when applied to action identifiers and to the conjunction connective when applied to formulas.
8. The semantic characterization of this construct is not uniform. [PEL 87b] and [PEL 87a] do not impose sequentiality contrary to [HAR 00]. Both differ on other aspects from the semantics in [GOL 89].

riched with this construct is defined taking the accessibility relation R to be a subset of $W \times 2^W$. The definitions affected by the introduction of $\wedge$ are (here $T, U, V \subseteq W$):

1) $\mathcal{M}, s \models \langle a \rangle \varphi$ if $\exists T\ (s, T) \in R(a)$ and $\mathcal{M}, t \models \varphi$ for all $t \in T$
2) $R(ab) = \{(s, T) \mid$
$\exists U\ (s, U) \in R(a)$ and $\forall u_i \in U\ \exists T_{u_i}(u_i, T_{u_i}) \in R(b)$ and $T = \bigcup T_{u_i}\}$
3) $R(a \wedge b) = \{(s, T) \mid$
$\exists U, V$ such that $(s, U) \in R(a), (s, V) \in R(b),$ and $T = U \cup V\}$

The meaning of $a \wedge b$ is captured in the logic by axiom

$$\langle a \wedge b \rangle \varphi \leftrightarrow \langle a \rangle \varphi \wedge \langle b \rangle \varphi$$

Although satisfactory in the context of concurrent programs, the introduction of this construct does not address our previous objections either. The milk-coffee example could be repeated here as well. Also, when several agents are acting concurrently, the correct result may depend on knowing who is doing what. For instance, an insurance company called to pay for a burned building is interested in knowing if the building really burned down and if the policy holder is responsible of arson. The relationship between agents and their actions is crucial when dealing with multi-agent systems.

The very features of multi-agent systems that are problematic in *PDL* intervene in shaping the more complex language of *mPDL*. In this language, the effects of an action could be altered by the other actions performed concurrently since the transition in the structure is determined by *all* executed actions. Furthermore, the agent and the action it executes are tied at the syntactic level by the position of the action identifiers in the modality. These two facts allow us to handle a wide class of situations for multi-agent systems and to provide a description of the system at a great level of detail.

To conclude, we highlight some important constraints that are expressible in our language. For the time being, let us assume $k = 2$, that is, that there are two agents in our multi-agent system (and that an ordering has been fixed). In our informal reading, formula $\begin{bmatrix} a \\ b \end{bmatrix} \varphi$ says that the actions denoted by a and b are executed concurrently. At the same time, it states that the action denoted by a is executed by the first agent and that the action denoted by b is executed by the other.

Let ϵ be the *null action*, that is, an action that corresponds to instruction "*do nothing*".[9] The rendition of the confluence axiom of dynamic logic in our multi-agent formalism allows us to capture subtle relationships

9. Informally, the *null action* is an action that does not alter the effects of other actions. As a consequence, the basic k-action that has the *null action* in each entry, is interpreted by the set of states $\{(s, s) | s \in W\}$.

$$\begin{bmatrix} a & \epsilon \\ \epsilon & b \end{bmatrix} \varphi \leftrightarrow \begin{bmatrix} \epsilon & a \\ b & \epsilon \end{bmatrix} \varphi \qquad \text{(execution-order does not matter)}$$

$$\begin{bmatrix} a & \epsilon \\ \epsilon & b \end{bmatrix} \varphi \leftrightarrow \begin{bmatrix} \epsilon & b \\ a & \epsilon \end{bmatrix} \varphi \qquad \text{(executing-agent does not matter)}$$

$$\begin{bmatrix} a & \epsilon \\ \epsilon & b \end{bmatrix} \varphi \leftrightarrow \begin{bmatrix} a & b \\ \epsilon & \epsilon \end{bmatrix} \varphi \qquad \text{(first agent suffices)}$$

$$\begin{bmatrix} a & b \\ \epsilon & \epsilon \end{bmatrix} \varphi \leftrightarrow \begin{bmatrix} b & a \\ \epsilon & \epsilon \end{bmatrix} \varphi \qquad \text{(linearly independent actions for agent 1)}$$

as well as other new constraints like

$$\begin{bmatrix} a & \epsilon \\ \epsilon & b \end{bmatrix} \varphi \leftrightarrow \begin{bmatrix} a & \epsilon \\ b & \epsilon \end{bmatrix} \varphi \qquad \text{(time independent actions.)}$$

3. Quantificational modalities

In the following pages we extend *mPDL* by introducing variables and quantifiers.

A simplifying assumption we make throughout this paper is that the agents in the system are homogeneous, that is, they have the same reasoning and memory capabilities, and skills. In a sense, these agents can be considered perfect clones of the same perfect reasoner.[10]

3.1. *Stuffing modalities with variables and quantifiers:* $\mathcal{L}_{\mathcal{MA}}$

As before, we fix an integer $k \in \mathbb{N}^+$ which informally stands for the number of agents in the system. We extend the syntax of the multi-agent propositional logic *mPDL* presented in section 2.2 as follows.

The language $\mathcal{L}_{\mathcal{MA}}$ (or $\mathcal{L}_{\mathcal{MA}(k)}$ if we need to make k explicit) uses three disjoint sets of basic identifiers:

- a non-empty countable set *PropId* of proposition identifiers
- a non-empty countable set *Var* of variables for actions
- a countable (possibly empty) set *ActId* of action identifiers

We will use $p, q, \ldots$ (possibly decorated) for proposition identifiers and, in a similar fashion, $x, y, \ldots$ for variables and $a, b, \ldots$ for action identifiers. For the sake of simplicity, neither functions nor relations are introduced, the 0-ary relations in *PropId* being the only exception.

10. This is mirrored in the semantics adopted in section 3.3.1. It is not a general constraint forced by the language itself.

Modality identifiers characterize the modalities of the language. These are particular matrices of action identifiers and (possibly quantified) variables. Definition 14 introduces these formally. (We use the term *quantifier* for the symbols $\forall$ and $\exists$ and *quantified variable* for the expressions $\forall x$ and $\exists x$.)

DEFINITION 14 (MODALITY IDENTIFIER IN $\mathcal{L}_{\mathcal{MA}(k)}$). —

A modality identifier *is a $k \times n$-matrix ($n \geq 1$)*

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{matrix}$$

where a_{ij} is either an action identifier, a variable, or a quantified variable.

We require that no variable occurs more than once in a modality identifier.

Two modality identifiers are said to be of same size *if they have the same number of columns (that is, the matrices have same size).*

A *term* in $\mathcal{L}_{\mathcal{MA}}$ is an action identifier or a variable. *Term* stands for the set of terms: $Term = ActId \cup Var$.

The set of $\mathcal{L}_{\mathcal{MA}(k)}$-formulas is the smallest set satisfying the following clauses:

a) all elements of $PropId$ are formulas (atomic formulas)

b) $\neg\varphi$ and $\varphi \rightarrow \psi$ are formulas if φ and ψ are formulas

c) $[M]\varphi$ is a formula if M is a modality identifier for $\mathcal{L}_{\mathcal{MA}(k)}$ and φ is a formula

The language we present has special features due to the form of the modality identifiers. For this, we now classify modal operators according to the identifiers associated with them.

DEFINITION 15 (MODAL OPERATORS FOR $\mathcal{L}_{\mathcal{MA}}$). —

A modal operator *or* modality *is an expression of the form $[M]$ where M is a modality identifier.*

1) A constant (modal) operator *is a modal operator whose entries are action identifiers. The set of constant operators is denoted by cOP.*

2) A variable (modal) operator *is a modal operator whose entries are action identifiers and variables, with at least one variable. The set of variable operators is denoted by vOP.*

3) A quantificational (modal) operator *is a modal operator whose entries are action identifiers, variables, and quantified variables, with at least one quantified variable. The set of quantificational operators is denoted by qOP.*

OP is the set of all modal operators in the language: $OP = cOP \cup vOP \cup qOP$

$$\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}; \qquad \begin{bmatrix} x & a_3 \\ a_2 & a_4 \end{bmatrix}; \qquad \begin{bmatrix} \forall x & z \\ a_2 & \exists v \end{bmatrix}$$

(a) (b) (c)

Figure 1. *Modal operators:* (a) *constant,* (b) *variable,* (c) *quantificational*

REMARK 16. — There is an obvious bijection between the set of modality identifiers and the set of modal operators in the language. We will use the two notions indifferently wherever this does not create ambiguity. For instance, we may write modality identifier M for the corresponding modal operator $[M]$ and vice versa.

We write $[M](i,j)$, or simply $M(i,j)$, for the entry (i,j) of M. For instance, $[M](2,1) = a$ means that the entry (2,1) of the modality identifier M (here seen as a matrix) contains action identifier a.

An entry containing the expression $\forall x_i$ ($\exists x_i$) is called an *universal* (*existential*) *entry*. Universal and existential entries are *quantified entries*. A *constant entry* is an entry containing an action identifier. Entries containing variables without quantifiers are called *parameter entries*.

The *scope of a modal operator* is the formula to which it is applied. The scope of a quantifier is the scope of the modal operator where it occurs. An occurrence of a variable x is said to be *bound* in a formula if it occurs quantified in a modal operator or lies within the scope of an operator where x occurs quantified. Otherwise, the occurrence is said to be *free*.
The set of *sentences* is the set of *closed formulas*, i.e., the set of formulas with no free occurrences of variables. In particular, all non-modal formulas are sentences. Also, an operator with a variable x in a parameter entry can occur in a sentence only within the scope of an operator that contains $\forall x$ or $\exists x$.

Here are some examples of logical formulas for $k = 2$ (formulas in the first row are open, formulas in the second row are sentences):

$$p_1 \rightarrow \begin{bmatrix} \forall x \\ y \end{bmatrix} p_2 \qquad ; \qquad \begin{bmatrix} \exists x & \exists y \\ a & \forall v \end{bmatrix} \begin{bmatrix} \forall x & z & b \\ b & \forall v & \exists u \end{bmatrix} p_1$$

$$\begin{bmatrix} a \\ b \end{bmatrix} p_1 \wedge \begin{bmatrix} a \\ \exists y \end{bmatrix} p_1 \wedge \neg \begin{bmatrix} \forall x \\ \forall y \end{bmatrix} p_2 \quad ; \quad \begin{bmatrix} a \\ \forall x \end{bmatrix} \left(\begin{bmatrix} x & b \\ \exists v & c \end{bmatrix} p_1 \leftrightarrow \neg \begin{bmatrix} \forall x \\ b \end{bmatrix} p_2 \right)$$

It should be clear that our modal operators bind less tightly than any other connective (analogously to the modalities in *PDL*, in *rPDL* and in *mPDL*).

Comparing modality identifiers is crucial to establish general properties of the language. For this goal, we devise a notation that allow us to modify modality identifiers entry by entry. The rest of this section is devoted to notational issues and modality constructs that will help us in this sense.

DEFINITION 17 (CHANGING MODALITY IDENTIFIERS). —

(i) Let $a \in ActId$.
Modality $[M_{(ij \leftarrow a)}]$ is like $[M]$ except that entry (i,j) of $[M_{(ij \leftarrow a)}]$ contains action identifier a.

(ii) Let $x \in Var$.
If x is new in $[M]$ or occurs, quantified or else, in $M(i,j)$, then $[M_{(ij \leftarrow x)}]$ is like $[M]$ except that entry (i,j) of $[M_{(ij \leftarrow x)}]$ contains the free variable x. Otherwise, $[M_{(ij \leftarrow x)}]$ and $[M]$ are the same modal operator.

(iii) Let $Q \in \{\forall, \exists\}$ and $x \in Var$.
If x is new in $[M]$ or x occurs, quantified or else, in $M(i,j)$, then $[M_{(ij \leftarrow Qx)}]$ is like $[M]$ except that entry (i,j) of $[M_{(ij \leftarrow Qx)}]$ contains Qx. Otherwise $[M_{(ij \leftarrow Qx)}]$ and $[M]$ are the same modal operator.

(iv) Let $\vec{u} = u_1, u_2, \ldots$ with u_i an action identifier, variable, or quantified variable. Then, $[M_{(i_1 j_1, i_2 j_2, \ldots \leftarrow u_1, u_2, \ldots)}]$ is like $[N_{(i_2 j_2, \ldots \leftarrow u_2, \ldots)}]$ where $[N]$ is the modality $[M_{(i_1 j_1 \leftarrow u_1)}]$. Alternatively, we write $[M_{(i_1 j_1, i_2 j_2, \ldots \leftarrow \vec{u})}]$.
Analogously, we write $[M_{(i_1 j_1, i_2 j_2, \ldots \leftarrow \forall \vec{u})}]$ for $[M_{(i_1 j_1, i_2 j_2, \ldots \leftarrow \forall u_1, \forall u_2, \ldots)}]$, and so for $[M_{(i_1 j_1, i_2 j_2, \ldots \leftarrow \exists \vec{u})}]$, provided all u_i are variables. If there is no danger of confusion, we adopt a more compact notation by writing $[M_{(\vec{x} \leftarrow \vec{u})}]\varphi$, $[M_{(\vec{x} \leftarrow \forall \vec{u})}]$, and $[M_{(\vec{x} \leftarrow \exists \vec{u})}]$ where $\vec{x}$ is a sequence of matrix-indices or variables occurring in the modality and $\vec{u}$ is as before.

EXAMPLES 18. — Let $[M] = \begin{bmatrix} \forall x & \exists y \\ a & \forall v \end{bmatrix}$, then:

$$[M_{(11 \leftarrow b)}] = \begin{bmatrix} b & \exists y \\ a & \forall v \end{bmatrix} \quad ; \quad [M_{(21 \leftarrow b)}] = \begin{bmatrix} \forall x & \exists y \\ b & \forall v \end{bmatrix}$$

$$[M_{(11 \leftarrow x)}] = \begin{bmatrix} x & \forall y \\ a & \forall v \end{bmatrix} \quad ; \quad [M_{(21 \leftarrow x)}] = [M]$$

$$[M_{(11 \leftarrow \exists x)}] = \begin{bmatrix} \exists x & \exists y \\ a & \forall v \end{bmatrix} \quad ; \quad [M_{(12 \leftarrow \exists x)}] = [M]$$

$$[M_{(22 \leftarrow \forall w)}] = \begin{bmatrix} \forall x & \exists y \\ a & \forall w \end{bmatrix} \quad ; \quad [(M_{(11,12 \leftarrow \exists w, \forall w)}] = \begin{bmatrix} \exists w & \exists y \\ a & \forall v \end{bmatrix}$$

DEFINITION 19 (UNIFORM MODALITY). —

Let M be a modality identifier.

(a) M is said to be $\forall$-uniform *if no entry is existential.*

(b) M is said to be $\exists$-uniform *if no entry is universal.*

(c) M is said to be uniform *if it is $\forall$-uniform and $\exists$-uniform.*

Equivalently, M is uniform if $M \in cOP \cup vOP$.

We conclude this part by introducing the notion of complementary modalities and the construct $\uplus$, called *merging*, that applies to them. This construct is quite important for the application of our formalism to multi-agent systems and will be used in section 4.1.2 to characterize the power of groups of agents.

DEFINITION 20 (COMPLEMENTARY MODALITIES AND MERGING). —

Let M, N be two modality identifiers of equal size.

(i) *M and N are said to be* complementary *if they satisfy all the followings:*

i.1) if $z \in Term$, then $M(i,j) = z$ if and only if $N(i,j) = z$;

i.2) if $M(i,j) = \forall x$, then $N(i,j) \in \{\forall x, \exists x\}$;

i.3) if $M(i,j) = \exists x$, then $N(i,j) = \forall x$;

and, symmetrically,

i.4) if $N(i,j) = \forall x$, then $M(i,j) \in \{\forall x, \exists x\}$;

i.5) if $N(i,j) = \exists x$, then $M(i,j) = \forall x$.

(ii) *Assume M, N are complementary.*

The merging of M and N *is the modality identifier $M \uplus N$ whose size is equal to the size of M (and N) and is defined by*

ii.1) $(M \uplus N)(i,j) = M(i,j)$, if $M(i,j)$ is an existential entry;

ii.2) $(M \uplus N)(i,j) = N(i,j)$, otherwise.

From the definition, the merging operator is symmetric, i.e., $M \uplus N = N \uplus M$.

The definition of complementarity among modalities says that to every existential entry in one of the identifiers corresponds an universal entry in the other one, and that the identifiers agree on the constant and parameter entries. The definition of $M \uplus N$ shows how to form a new operator mixing the entries of two complementary operators giving priority to existential entries.

Here is an example of the application of these notions in a 3-agent system. Let $[M] = \begin{bmatrix} \exists x & \forall y \\ a & \forall u \\ \forall v & z \end{bmatrix}$ and $[N] = \begin{bmatrix} \forall x & \forall y \\ a & \exists u \\ \exists v & z \end{bmatrix}$ be two operators. Clearly M and N are complementary. Then, $[M \uplus N]$ is defined and we have

$$[M \uplus N] \quad = \quad \begin{bmatrix} \exists x & \forall y \\ a & \exists u \\ \exists v & z \end{bmatrix}$$

3.2. *Structures for* $\mathcal{L}_{\mathcal{MA}}$

The semantics of formulas without variables is naturally inherited from the propositional language *mPDL*. Formulas with variables (with or without quantifiers) are new and are interpreted through sets of transitions on Kripke structures.

Generally speaking, to evaluate an operator one needs to associate each entry of the modality identifier with an action. This relationship between entries and actions is immediate for the constant operators and in the case of variable operators it can be provided by the *environment function* (here a function $\Im : Var \to Act$) thus following the semantics of first-order logic (*FOL*). The case of quantificational operators requires further assumptions since, as we shall see, it depends on the quantifiers as well as on the *internal structure* of the operator's identifier.

Since in our modalities quantified variables are not organized in a linear order, we cannot apply the method of *FOL* to isolate instances of these modalities unless an ordering of the entries is imposed somehow. Indeed, first-order logic deals only with sequentially displayed quantifiers. Before proposing a way out, let us see how one can apply previous ideas for the formulas without quantifiers. The fundamental notions stated for *mPDL* are adopted in $\mathcal{L}_{\mathcal{MA}}$ without change, in particular: the notions of of k*-action*, *multi-agent Kripke frame*, and *multi-agent Kripke structure*, see Definitions 9, 10, 11.

Following *mPDL* and in contrast to standard multi-relational Kripke frames, single actions do not identify transitions in the frame. Instead, transitions are associated with particular columns of actions. For the sake of clarity, we provide here the definition of multi-agent Kripke structure for $\mathcal{L}_{\mathcal{MA}}$.

DEFINITION 21 (MULTI-AGENT KRIPKE STRUCTURE FOR $\mathcal{L}_{\mathcal{MA}}$). —

A Multi-agent Kripke Structure for $\mathcal{L}_{\mathcal{MA}(k)}$ *is a 4-tuple* $\mathcal{M} = \langle W, Act; R, [\![\cdot]\!]\rangle$ *where:*

1) $\langle W, Act; R\rangle$ *is a multi-agent Kripke frame for mPDL and the index* k*, and*

2) $[\![\cdot]\!]$ *is a function* (the valuation function) *such that* $[\![p]\!] \subseteq W$ *for* $p \in PropId$ *and* $[\![a]\!] \in Act$ *for* $a \in ActId$.

As before, given a multi-agent Kripke structure $\mathcal{M}$, $[\![\cdot]\!]$ is extended to all constant operators in the language by

$$\left[\!\!\left[\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{matrix}\right]\!\!\right] =_{def} \begin{matrix} [\![a_{11}]\!] & [\![a_{12}]\!] & \ldots & [\![a_{1n}]\!] \\ [\![a_{21}]\!] & [\![a_{22}]\!] & \ldots & [\![a_{2n}]\!] \\ \vdots & \vdots & & \vdots \\ [\![a_{k1}]\!] & [\![a_{k2}]\!] & \ldots & [\![a_{kn}]\!] \end{matrix}$$

In standard modal logic, a Kripke frame for the propositional language is a Kripke frame for the first-order modal language also. Analogously, a multi-agent Kripke frame for *mPDL* is a multi-agent Kripke frame for the quantificational language corresponding to it. However, these two types of frame differ on a crucial aspect. In standard Krikpe frames, the domain of quantification is independent from the frame itself (one can augment the frame with any set as domain of quantification). On the contrary, in multi-agent Kripke frames it is the very set of actions *Act* that plays the role of domain of quantification. Also, sequences of elements in *Act* (the basic k-actions) provide the labels for the transitions in the frame. Thus, in multi-agent Kripke frames there is an explicit tie between the quantification domain and the transitions. Finally, note that the extended frame has constant domain of quantification since the domain *Act* is the same in all possible states.

In the remaining of the paper, we use Kripke frames (structures) to refer to multi-agent Kripke frames (structures).

3.3. *Semantics rises to the occasion*

In extending the syntax of *mPDL* to the richer syntax of $\mathcal{L}_{\mathcal{MA}}$, we have been working within a path well-known to logicians, i.e., the introduction of variables and quantifiers. The whole process has been quite smooth, and so was the introduction of frames for the latter language. However, the semantic interpretation of the resulting system is not straightforward.

Once the environment $\Im : Var \to Act$ is fixed, one can extend the valuation function $[\![\cdot]\!]$ over variable modal operators.

DEFINITION 22 (VALUATION OVER *vOP*). —

Given a multi-agent Kripke structure $\langle W, Act; R, [\![\cdot]\!]\rangle$ and an environment $\Im$,

i) if x is a variable, then put $[\![x]\!] =_{def} \Im(x)$,

ii) if A is an operator in vOP, then put

$$\left[\!\!\left[\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{matrix}\right]\!\!\right] =_{def} \begin{matrix} [\![a_{11}]\!] & [\![a_{12}]\!] & \dots & [\![a_{1n}]\!] \\ [\![a_{21}]\!] & [\![a_{22}]\!] & \dots & [\![a_{2n}]\!] \\ \vdots & \dots & & \vdots \\ [\![a_{k1}]\!] & [\![a_{k2}]\!] & \dots & [\![a_{kn}]\!] \end{matrix}$$

(We have further overloaded the notation for the valuation function. Since the sets *PropId*, *ActId*, and *Var* are disjoint, this abuse of notation causes no confusion.)

In Definition 17 we have shown how to represent modality identifiers which differ on some entries only. Here we give the corresponding definition for k-actions.

DEFINITION 23 (CHANGING k-ACTIONS). —

Given a set Act of actions, let A be a k-action and $\alpha, \beta, \ldots \in Act$, then:

(i) $A_{(ij\leftarrow\alpha)}$ is k-action A except that $A_{(ij\leftarrow\alpha)}(i,j) = \alpha$.

(ii) $A_{(i_1j_1,i_2j_2,\ldots\leftarrow\alpha,\beta,\ldots)}$ is k-action $(\ldots((A_{(i_1j_1\leftarrow\alpha)})_{(i_2j_2\leftarrow\beta)}),\ldots)$.

(iii) $A_{(i_1 j_1, i_2 j_2, \ldots \leftarrow \vec{\alpha})}$ *is* k*-action* $(\ldots((A_{(i_1 j_1 \leftarrow \alpha_1)})_{(i_2 j_2 \leftarrow \alpha_2)}), \ldots)$.

DEFINITION 24 (INSTANCES OF A MODALITY). —

Given a Kripke structure $\langle W, Act; R, [\![\cdot]\!]\rangle$ *and an environment* $\Im$*, a* k*-action* A *is an* instance of $[M]$ *(equivalently,* instance of modality identifier M*) if* A *is obtained from* M *by substituting:*

a) an action in Act for each quantified variable

b) $[\![a]\!]$ *for each element* $a \in Term$*.*

We write Inst(M) *for the set of instances of* $[M]$*.*

Action $A(i,j)$ *is said to* instantiate variable x of M *if* $A \in Inst(M)$ *and* x *occurs, quantified or else, at* $M(i,j)$*.*

PROPOSITION 25. — *Given a structure* $\langle W, Act; R, [\![\cdot]\!]\rangle$*, an environment* $\Im$*, and an action identifier* a*,*

$$Inst(M_{(ij \leftarrow a)}) = \{A_{(ij \leftarrow \alpha)} \mid A \in Inst(M) \text{ and } [\![a]\!] = \alpha\}.$$

PROPOSITION 26. — *Given a structure* $\mathcal{M}$ *with* $|Act| > 1$ *and environment* $\Im$*,*

$$|Inst(M)| = 1 \text{ if and only if } [M] \notin qOP.$$

DEFINITION 27 (CHANGING THE ENVIRONMENT). —

Fix an environment function $\Im$*. We write* $\Im^{a_1 \, \cdots \, a_r}_{y_1 \, \cdots \, y_r}$ *(with* $y_i \neq y_j$*) for the environment function defined by:*

$\Im^{a_1 \, \cdots \, a_r}_{y_1 \, \cdots \, y_r}(x) = \Im(x)$ *for* $x \notin \{y_1, \ldots, y_r\}$*,*
$\Im^{a_1 \, \cdots \, a_r}_{y_1 \, \cdots \, y_r}(y_i) = a_i$*.*

Given a k*-action* A *and a modal operator* $[M]$ *with* $A \in$ *Inst(M), we write* $\Im^A_M$ *for the environment function defined as follows:*

$\Im^A_M(y) = \Im(y)$ *for* y *not occurring in* M*,*
$\Im^A_M(y) = A(i,j)$ *for* y *occurring (quantified or else) at* $M(i,j)$*.*

Note that $\Im^A_M$ is well defined since, from Definition 14, a variable can occur at most once in the modality identifier M.

3.3.1. *Sequential semantics:* $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$

In this section, we present a semantics for $\mathcal{L}_{\mathcal{MA}}$ that is suitable for multi-agent systems. Generally speaking, in modal logic one can easily define different semantics for a language; the comparison of different interpretations of the language $\mathcal{L}_{\mathcal{MA}}$ is an important aspect of our work. However, here we disregard this issue and concentrate on one interpretation of the language only. The interpreted language we obtain is dubbed $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ (or $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ if we need to make the index k explicit.)

Fix a Krikpe structure $\mathcal{M}$ and a state *s*. Let $\Im$ be an environment.
We write $\mathcal{M}, s, \Im \models_S \varphi$ to mean that, in the semantics $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$, the formula φ is *true* (equivalently, *satisfied*) at state *s* of structure $\mathcal{M}$ for environment $\Im$. We write $\mathcal{M}, s \models_S \varphi$ if φ is true for any environment $\Im$. Furthermore, $\mathcal{M} \models \varphi$ means that formula φ is *valid in* $\mathcal{M}$, that is, it is true at each state of $\mathcal{M}$.

Relation $\models_S$ is defined recursively as follows:

1_S) Let $p \in PropId$, then $\mathcal{M}, s, \Im \models_S p$ if $s \in [\![p]\!]$

2_S) $\mathcal{M}, s, \Im \models_S \neg\varphi$ if $\mathcal{M}, s, \Im \not\models_S \varphi$

3_S) $\mathcal{M}, s, \Im \models_S \varphi \rightarrow \psi$ if $\mathcal{M}, s, \Im \not\models_S \varphi$ or $\mathcal{M}, s, \Im \models_S \psi$

4_S) Let $[M]$ be uniform, then
$\mathcal{M}, s, \Im \models_S [M]\varphi$ if for all s' such that $(s, s') \in R([\![M]\!])$, then $\mathcal{M}, s', \Im \models_S \varphi$

5_S) Let $\vec{x}$ be all the existentially quantified variables in M, then,
$\mathcal{M}, s, \Im \models_S [M]\varphi$ if there exists a sequence $\vec{\alpha}$ of elements in *Act* (not necessarily distinct) such that if $A \in Inst(M_{(\vec{x} \leftarrow \vec{\alpha})})$, then for all s' such that $(s, s') \in R(A)$, we have $\mathcal{M}, s', \Im^A_M \models_S \varphi$

As anticipated, we dub $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ the language $\mathcal{L}_{\mathcal{MA}}$ with the semantics given by 1_S) – 5_S). Note that clause 4_S) is consistent with the semantics of *mPDL* and is a particular case of clause 5_S). The latter will be motivated below.

DEFINITION 28 (MULTI-AGENT KRIPKE MODEL IN $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$). —

A multi-agent Kripke model *for a set of formulas* Σ *in* $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ *is a structure* $\mathcal{M}$ *for* $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ *such that all formulas* $\varphi \in \Sigma$ *are valid in* $\mathcal{M}$.

Having stated the semantics of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$, it is now evident that our reading of quantified variables depart from their standard meaning. In the modality identifiers, we write $\forall x$ (and $\exists x$) to identify both a quantified variable and an occurrence of that very variable. In this case, there is no danger of confusion. Nonetheless, it is advisable to adopt a more explicit notation, for instance $\forall x.x$ and $\exists x.x$, when considering extensions of the language $\mathcal{L}_{\mathcal{MA}}$ that allow for more complex combinations of quantifiers and terms in the entries of the modality identifiers.

We remark that the semantics of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ is entirely first-order. Although clause 5_S) quantifies over operators, this should be taken as a figure of speech and not as an ontological requirement. We have and will freely use expressions like “any instance A of $[M]$" as a shorthand for “any k-action corresponding to arbitrary instantiations of the universally quantified variables occurring in $[M]$.” For the sake of clarity, in $5'_S$) we rewrite clause 5_S) highlighting the role of universal quantifiers and avoiding quantifying over operators. According to Definition 24, $5'_S$) is equivalent to 5_S).

$5'_S$) Let $[M] \in qOP$, $\vec{x}$ all the existentially quantified variables, and $\vec{y}$ all the universally quantified variables in M (the order they occur is irrelevant). Then, $\mathcal{M}, s, \Im \models_S [M]\varphi$ if there is a sequence $\vec{\alpha}$ of elements in *Act*, such that for any sequence $\vec{\beta}$ also of elements in *Act*, for $B = [\![M_{(\vec{x},\vec{y} \leftarrow \vec{\alpha},\vec{\beta})}]\!]$ and $(s, s') \in R(B)$, we have $\mathcal{M}, s', \Im^B_M \models_S \varphi$.

But what is the rationale of clause 5_S)? Clearly, one expects $[M]\varphi$ to be true when for each instance A of M, formula $[A]\varphi$ is true. This fits with clause 5_S) above. However, this clause requires much less to conclude that a formula is true. According to clause 5_S), $[M]\varphi$ is true if there are actions such that, when used as interpretations of the existentially quantified variables of M, any instance B of M (that respects this interpretation) satisfies $\mathcal{M}, s, \Im^B_M \models_S [M']\varphi$ according to clause 4_S). (Below we give an example in MAS to motivate this choice). Thus, in this semantics all existentially quantified variables are instantiated before the universally quantified ones. This constraint forces a formula in $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ to be true only when, once the values for the existentially quantified variables is fixed, one cannot reach "undesired" states by varying the value of universally quantified variables. From the point of view of a multi-agent system, a formula $[M]\varphi$ is true if the agents have a way to force φ (i.e, to force instances of the quantificational operator that take to those states where φ is true) by deciding beforehand which actions to execute at the existential positions and without need to know the actions executed at the universal positions on M (and thus their possible interferences.)

Such a system is suited for describing a (restricted) notion of *plan*. For instance, let Alan and Bill be coworkers in a project for their company and suppose they have to complete the project by the end of the day. They might need to work on it at different stages of the development and, as usual in a social environment, they have to combine their work with other commitments as well. The first thing they can do is to develop a plan taking into account their constraints and the project needs. Suppose Bill is at a meeting early in the morning and that Alan has a doctor visit before lunch. After lunch they are both free but, in the mid afternoon Alan has to meet with the office manager. A plan for the project is any combination of actions that are compatible with the given constraints and that guarantee the completion of the project by the due time. Once such a plan is selected (if it exists), its application ensures the success of the coworkers' efforts.

In our language, the following formula describes this situation and one can use it to isolate such a plan (here Alan is agent **1** and Bill is agent **2**):

$$\begin{bmatrix} \exists x_1 & d & a & \exists x_4 & \forall x_5 \\ \forall y_1 & \exists y_2 & a & \exists y_4 & \exists y_5 \end{bmatrix} \varphi$$

where φ stands for "the project is finished". Here variable y_1 is universally quantified since it refers to a time-slot in which Bill will do what asked by the manager. Similarly for variable x_5, this time with respect to Alan. The second time-slot for Alan contains an action identifier, d, which denotes the action "go to the doctor". Similarly, the third time-slot (for both the agents) contains an action identifier, a, corresponding to "have lunch". The remaining variables are existentially quantified so that Alan and Bill can decide what actions to perform at those positions in order to complete the project in time.

By choosing beforehand the instances of variables x_1, x_4, y_2, y_4, and y_5 the agents define a plan which states what to do when they work on the project. $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ classifies the formula as true is and only if there exists a choice of instances for x_1, x_4, y_2, y_4, and y_5 that guarantees the completion of the project. However, if such a choice cannot be successfully made without any knowledge of the values for y_1 or x_5, then the formula is false according to $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$. That is, if in the early morning their director asks them if they can finish the project in time, they will answer "yes" in the first case, "no' (that is, "we cannot guarantee it") in the latter.

In particular, note that the semantics we have described does not consider adaptable plans like, for instance, plans that include "$if \dots then$" conditions. A semantics applying a more general notion of plan is necessarily more complicated and may require game-theoretical notions. See [BOR 03] for an example. Finally, although in this paper we allow for simple expressions only in the modal identifiers, like $\exists x$ and $\forall x$ with $\exists, \forall$ resembling the traditional universal and existential quantifiers, the application of a richer class of expressions may become necessary for modeling some multi-agent systems. For instance, expressions like $\exists_{\varphi ?} x$ with meaning "choose an item x knowing the truth-value of φ" or $\exists_{Bel(\varphi)} x$ with meaning "choose an item x assuming that φ is true".

3.3.2. *$\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ at work*

We give a few examples to make clear how the semantics just introduced works. Also, these examples clarify our claim that the description of multi-agent systems is an important motivation for this logic system.

In order to keep things simple, we consider a structure $\mathcal{M}_{\mathcal{S}}$ where $k = 2$ and $W = \{s_0, s_1, s_2\}$. For actions we take $Act = \{\alpha, \beta\}$. The structure is pictured in Figure 2. (s_0 is the state at which the formulas are evaluated.)

We consider just one proposition identifier p whose valuation is $[\![p]\!]_S = \{s_1\}$ and we evaluate at s_0 three formulas:

$$\begin{bmatrix} \exists x & \exists z \\ \forall y & \forall u \end{bmatrix} p\ , \quad \begin{bmatrix} \exists x & \forall z \\ \forall y & \exists u \end{bmatrix} p\ , \quad \begin{bmatrix} \forall x & \exists z \\ \exists y & \forall u \end{bmatrix} p\ .$$

According to the semantic clause 5_S) for $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ (more precisely, $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(2)}$) formula $\begin{bmatrix} \exists x & \exists z \\ \forall y & \forall u \end{bmatrix} p$ is true in $\mathcal{M}_{\mathcal{S}}$ at s_0 if there are values for x and z such that, for any instantiation of y and u, the 2-action obtained through substituting these values for x, z, y, u (respectively) brings the system to state s_1. False, otherwise.

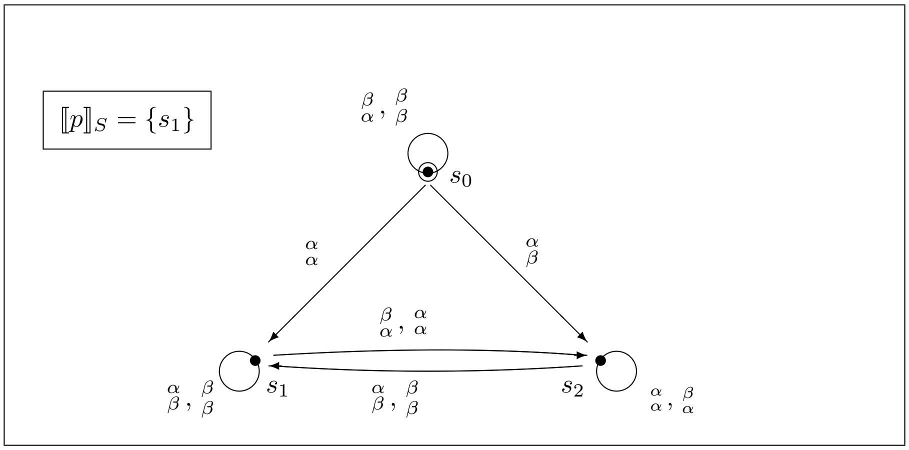

Figure 2. *The two-agent structure* $\mathcal{M}_{\mathcal{S}} = \langle \{s_0, s_1, s_2\}, \{\alpha, \beta\}; R, [\![\cdot]\!]_S \rangle$ *with* R *and* $[\![p]\!]_S$ *as shown.*

In practice, the different choices of values for x and z isolate (disjoint) subsets of instances (2-actions) for the quantificational operator as follows (we write $x, z \hookleftarrow \alpha, \beta$ for: x is instantiated by α and z by β; the 2-actions in a set are obtained by taking any possible value for y and u):

$$\begin{aligned}
x, z \hookleftarrow \alpha, \alpha \quad &: \quad \left\{ \begin{matrix}\alpha & \alpha \\ \alpha & \alpha\end{matrix}, \begin{matrix}\alpha & \alpha \\ \alpha & \beta\end{matrix}, \begin{matrix}\alpha & \alpha \\ \beta & \alpha\end{matrix}, \begin{matrix}\alpha & \alpha \\ \beta & \beta\end{matrix} \right\} \\
x, z \hookleftarrow \alpha, \beta \quad &: \quad \left\{ \begin{matrix}\alpha & \beta \\ \alpha & \alpha\end{matrix}, \begin{matrix}\alpha & \beta \\ \alpha & \beta\end{matrix}, \begin{matrix}\alpha & \beta \\ \beta & \alpha\end{matrix}, \begin{matrix}\alpha & \beta \\ \beta & \beta\end{matrix} \right\} \\
x, z \hookleftarrow \beta, \alpha \quad &: \quad \left\{ \begin{matrix}\beta & \alpha \\ \alpha & \alpha\end{matrix}, \begin{matrix}\beta & \alpha \\ \alpha & \beta\end{matrix}, \begin{matrix}\beta & \alpha \\ \beta & \alpha\end{matrix}, \begin{matrix}\beta & \alpha \\ \beta & \beta\end{matrix} \right\} \\
x, z \hookleftarrow \beta, \beta \quad &: \quad \left\{ \begin{matrix}\beta & \beta \\ \alpha & \alpha\end{matrix}, \begin{matrix}\beta & \beta \\ \alpha & \beta\end{matrix}, \begin{matrix}\beta & \beta \\ \beta & \alpha\end{matrix}, \begin{matrix}\beta & \beta \\ \beta & \beta\end{matrix} \right\}
\end{aligned}$$

Recall that we have

$$R\begin{pmatrix}\alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4\end{pmatrix} = R\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} \circ R\begin{pmatrix}\alpha_3 \\ \alpha_4\end{pmatrix}$$

It is then simple to verify that in each of the sets above there is some instance for $\begin{bmatrix}\exists x & \exists z \\ \forall y & \forall u\end{bmatrix}$ by which the system reaches state s_0 or s_2. Indeed, it is not possible to find instances for x and z such that the system always ends up at s_1 (the only state where p is true). Consequently, the formula is not true in $\mathcal{M}_{\mathcal{S}}$ at s_0.

In our informal reading of the formalism, the first agent cannot devise a successful plan (choice for the existentially quantified variable) that forces the system to make p true independently from what the other agent does.

Now, consider the second formula, $\begin{bmatrix} \exists x & \forall z \\ \forall y & \exists u \end{bmatrix} p$.
Analogous to the previous case, this formula is true in $\mathcal{M}_{\mathcal{S}}$ at s_0 if there are values for x and u such that, for any instantiation of y and z, the 2-actions obtained through substituting these values for x, u, y, z (respectively) all bring the system to state s_1. False, otherwise.

Here are the sets of instances determined in this way.

$$x, u \hookleftarrow \alpha, \alpha \quad : \quad \left\{ \begin{matrix} \alpha & \alpha \\ \alpha & \alpha \end{matrix} , \begin{matrix} \alpha & \beta \\ \alpha & \alpha \end{matrix} , \begin{matrix} \alpha & \alpha \\ \beta & \alpha \end{matrix} , \begin{matrix} \alpha & \beta \\ \beta & \alpha \end{matrix} \right\}$$

$$x, u \hookleftarrow \alpha, \beta \quad : \quad \left\{ \begin{matrix} \alpha & \alpha \\ \alpha & \beta \end{matrix} , \begin{matrix} \alpha & \beta \\ \alpha & \beta \end{matrix} , \begin{matrix} \alpha & \alpha \\ \beta & \beta \end{matrix} , \begin{matrix} \alpha & \beta \\ \beta & \beta \end{matrix} \right\}$$

$$x, u \hookleftarrow \beta, \alpha \quad : \quad \left\{ \begin{matrix} \beta & \alpha \\ \alpha & \alpha \end{matrix} , \begin{matrix} \beta & \beta \\ \alpha & \alpha \end{matrix} , \begin{matrix} \beta & \alpha \\ \beta & \alpha \end{matrix} , \begin{matrix} \beta & \beta \\ \beta & \alpha \end{matrix} \right\}$$

$$x, u \hookleftarrow \beta, \beta \quad : \quad \left\{ \begin{matrix} \beta & \alpha \\ \alpha & \beta \end{matrix} , \begin{matrix} \beta & \beta \\ \alpha & \beta \end{matrix} , \begin{matrix} \beta & \alpha \\ \beta & \beta \end{matrix} , \begin{matrix} \beta & \beta \\ \beta & \beta \end{matrix} \right\}$$

All the 2-actions obtained from assignment $x, u \hookleftarrow \alpha, \beta$ bring the system to state s_1 as desired. According to the semantics, the formula is true at s_0.

In our informal reading, the two agents can devise *in collaboration and beforehand* a successful plan (choice for the existentially quantified variable) that forces the system to make p true no matter what they do when they are not constrained by the plan.

Finally, consider formula $\begin{bmatrix} \forall x & \exists z \\ \exists y & \forall u \end{bmatrix} p$, which is obtained from the previous by substituting the quantifiers by their duals. We have

$$y, z \hookleftarrow \alpha, \alpha \quad : \quad \left\{ \begin{matrix} \alpha & \alpha \\ \alpha & \alpha \end{matrix} , \begin{matrix} \alpha & \alpha \\ \alpha & \beta \end{matrix} , \begin{matrix} \beta & \alpha \\ \alpha & \alpha \end{matrix} , \begin{matrix} \beta & \alpha \\ \alpha & \beta \end{matrix} \right\}$$

$$y, z \hookleftarrow \alpha, \beta \quad : \quad \left\{ \begin{matrix} \alpha & \beta \\ \alpha & \alpha \end{matrix} , \begin{matrix} \alpha & \beta \\ \alpha & \beta \end{matrix} , \begin{matrix} \beta & \beta \\ \alpha & \alpha \end{matrix} , \begin{matrix} \beta & \beta \\ \alpha & \beta \end{matrix} \right\}$$

$$y, z \hookleftarrow \beta, \alpha \quad : \quad \left\{ \begin{matrix} \alpha & \alpha \\ \beta & \alpha \end{matrix} , \begin{matrix} \alpha & \alpha \\ \beta & \beta \end{matrix} , \begin{matrix} \beta & \alpha \\ \beta & \alpha \end{matrix} , \begin{matrix} \beta & \alpha \\ \beta & \beta \end{matrix} \right\}$$

$$y, z \hookleftarrow \beta, \beta \quad : \quad \left\{ \begin{matrix} \alpha & \beta \\ \beta & \alpha \end{matrix} , \begin{matrix} \alpha & \beta \\ \beta & \beta \end{matrix} , \begin{matrix} \beta & \beta \\ \beta & \alpha \end{matrix} , \begin{matrix} \beta & \beta \\ \beta & \beta \end{matrix} \right\}$$

One can verify that all sets above contain an instance of the operator for which the formula is not true in $\mathcal{M}_{\mathcal{S}}$ at s_0 (it is enough to check the very first 2-action in each set). We conclude that the formula is false.

In our informal reading, this result tells us that if the two agents behave with opposite intentions with respect to the previous formula, their collaboration does not suffice to grant the existence of a successful plan.

The model $\mathcal{M}_{\mathcal{S}}$ of Figure 2 allows us to prove an interesting property.

PROPOSITION 29. — *There exist modal operators M, N in $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ such that*

$$[MN]\varphi \not\equiv [M][N]\varphi$$

However, for A, B uniform, we have

$$[AB]\varphi \equiv [A][B]\varphi$$

PROOF. — An example for the first claim is provided by the model $\mathcal{M}_{\mathcal{S}}$ of Figure 2 by taking $M = \frac{\forall x}{\exists u}$, $N = \frac{\exists z}{\exists u}$, and $\varphi = p$.

The second claim follows from the semantics of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$. ■

3.4. *Which constructs for $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$?*

Now that the interpreted language has been introduced and motivated, we spend a few words on different constructs that might be of interest for enriching the language.

Although in this work we have excluded constructs on action identifiers in order to keep the language simple, in the light of section 2 it is natural to look at those provided by *PDL*. The adoption of *PDL* constructs can be made at two levels. Take the "non-deterministic choice" operator $\cup$: one can add it at the *operator-level*, for instance writing complex modalities like $\begin{bmatrix} a & \cup & b \\ c & & d \end{bmatrix}$ where $\cup$ applies to columns in modal operators, or at the *action-level*, like in $\begin{bmatrix} a \cup b \\ c \end{bmatrix}$. Analogously, the introduction of the "star" operator $*$ would allow us to write formulas of form $\left[\begin{pmatrix} a \\ c \end{pmatrix}^{*}\right]$ in the first case, and formulas of form $\begin{bmatrix} a^{*} \\ c \end{bmatrix}$ in the other case. These formulas have different imports in the language. The first kind of formulas does not affect the action-synchronicity implicit in $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ and so it would be easy to introduce. Instead, formulas with $*$ (or even the composition construct) at the entry level open the way to the description of asynchronous systems and requires some changes in the semantics. In both cases, the interaction of the new constructs with the quantifiers needs to be carefully considered.

An interesting issue is the inclusion of the "test" construct (at the entry level) and its interaction with the existential quantifier. The introduction of a construct based on "test" has been suggested at the end of section 3.3.1 although it is quite different from the operator of *PDL*. In our reading of the modalities, we could allow an expression of form $\exists_{\varphi?}x$ in a modality $[M]$ with the meaning "choose a value for x knowing the truth-value of φ at the moment in which x has to be performed". Note that the semantics of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ makes the result of the test publicly available to all the agents. By tying strictly the result of the test with the action chosen at that point by the very agent performing the test, we obtain an operator that fits our overall framework and is easily motivated in the area of multi-agent systems. Indeed, it would allow to describe more complex plans in the language (essentially "*if...then*" conditions) and seems to be worth to study.

4. The logic of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$

4.1. *Deductive system*

The substitution of variables by terms is a crucial process in formal languages. In $\mathcal{L}_{\mathcal{MA}}$, this substitution takes place only in modalities since variables do not occur outside these operators. If modal operators are explicitly given, one can use the notation of Definition 17. For the other cases, especially when there are nested modalities, we need to represent the substitution of variables without listing every operator involved. For this reason, we refer to some standard notation. Other definitions are here introduced as well.

DEFINITION 30 (SUBSTITUTION OF TERMS). —

Let $\vec{x} = x_1, \ldots, x_n$ be a sequence of terms (not necessarily distinct) and $\vec{y} = y_1, \ldots, y_n$ a sequence of distinct variables. Given a formula φ, $\{\vec{x}/\vec{y}\}\varphi$ is the formula obtained from φ by the simultaneous substitution of the free occurrences of y_i by x_i $(1 \leq i \leq n)$.

DEFINITION 31 (FREE FOR A VARIABLE). —

Let $x, y \in Var$. x is free for *y in formula φ if no free occurrence of y in φ is in the scope of $\forall x$ or $\exists x$.*

Also, we need a couple of definitions on subformulas.

DEFINITION 32 (POSITIVE AND NEGATIVE POSITION OF SUBFORMULAS). —

Given a formula ψ, an occurrence $\tilde{\varphi}$ of a subformula φ of ψ is said to be positive or negative according to the following conditions:

i) $\tilde{\varphi}$ is positive in φ

ii) if $\tilde{\varphi}$ is positive (negative) in γ, then

$\tilde{\varphi}$ is positive (negative) in $\chi \rightarrow \gamma$

$\tilde{\varphi}$ is positive (negative) in $[M]\gamma$

$\tilde{\varphi}$ is negative (positive) in $\gamma \rightarrow \chi$

$\tilde{\varphi}$ is negative (positive) in $\neg\gamma$

Also, given a positive (negative) occurrence of $\tilde{\varphi} = [M]\chi$ in ψ, we say that the corresponding occurrence of $[M]$ is positive (negative) as well.

DEFINITION 33 (CHANGING MODAL SUBFORMULAS). —

Let $[M]$ be a modal operator and $\vec{x} = x_1, \ldots, x_p$ a list of terms with x_1 occurring (perhaps quantified) in $M(i_1, j_1)$, ... , x_p occurring (perhaps quantified) in $M(i_p, j_p)$. Let φ be $[M]\chi$ and $\vec{z} = z_1, \ldots, z_p$ be new in all modalities of φ, then we write

i) $\varphi_{(\vec{x}\leftarrow\vec{z})}$ *for* $[M_{(i_1j_1,\ldots,i_pj_p\leftarrow\vec{z})}]\{\vec{z}/\vec{x}\}\chi$,

ii) $\varphi_{(\vec{x}\leftarrow\forall\vec{z})}$ *for* $[M_{(i_1j_1,\ldots,i_pj_p\leftarrow\forall\vec{z})}]\{\vec{z}/\vec{x}\}\chi$,

iii) $\varphi_{(\vec{x}\leftarrow\exists\vec{z})}$ *for* $[M_{(i_1j_1,\ldots,i_pj_p\leftarrow\exists\vec{z})}]\{\vec{z}/\vec{x}\}\chi$.

Let ψ be a formula, $[M]\chi$ a subformula of ψ, and $\tilde{\varphi}$ one single occurrence of $[M]\chi$ in ψ, then we write

i) $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\vec{z})}\}$ *for ψ with $\tilde{\varphi}$ substituted by an occurrence of* $\varphi_{(\vec{x}\leftarrow\vec{z})}$;

ii) $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{z})}\}$ *for ψ with $\tilde{\varphi}$ substituted by an occurrence of* $\varphi_{(\vec{x}\leftarrow\forall\vec{z})}$;

iii) $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\exists\vec{z})}\}$ *for ψ with $\tilde{\varphi}$ substituted by an occurrence of* $\varphi_{(\vec{x}\leftarrow\exists\vec{z})}$.

Furthermore, $\psi\{\tilde{\varphi}^1_{(\vec{x}_1\leftarrow\vec{z}_1)}, \ldots, \tilde{\varphi}^n_{(\vec{x}_n\leftarrow\vec{z}_n)}\}$, with occurrences $\tilde{\varphi}^i$ and $\tilde{\varphi}^j$ distinct for $i \neq j$, is obtained recursively by $(\ldots(\psi\{\tilde{\varphi}^1_{(\vec{x}_1\leftarrow\vec{z}_1)}\})\ldots)\{\tilde{\varphi}^n_{(\vec{x}_n\leftarrow\vec{z}_n)}\}$.

Let us recall (and expand) our notation on provable formulas (see page 6.)

DEFINITION 34. —

Let Λ be a set of axioms closed with respect to a set of deduction rules, then

(i) $\vdash_\Lambda \varphi$ *stands for* $\varphi \in \Lambda$;

(ii) Let Σ be a set of formulas. We write $\Sigma \vdash_\Lambda \varphi$ to mean that there exist $\psi_1, \ldots, \psi_n \in \Sigma$ such that $(\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \varphi \in \Lambda$.

As before, when there is no danger of confusion, we omit the index Λ in $\vdash$.

A logic for $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ is the smallest set of formulas of $\mathcal{L}_{\mathcal{MA}}$ (for the fixed k) closed under the rules listed below and containing all the instances of the following schemas ($[M], [N]$ range over all modal operators in the language).

Axiom schemas:

(PL) All instances of propositional tautologies in $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$

(K^S) For $[M]$ $\forall$-uniform
$[M](\varphi \rightarrow \psi) \rightarrow ([M]\varphi \rightarrow [M]\psi)$ (Normality)

$(\exists I^S)$ For $M(i, j) = x \in Var$ and a in *ActId* or a equal to x
$[M_{(ij\leftarrow a)}]\{a/x\}\varphi \rightarrow [M_{(ij\leftarrow\exists y)}]\{y/x\}\varphi$ (Existential Introduction)
where either y is x or y has no free occurrences in φ, is free for x in φ, and is new in M

($\forall E^S$) For $M(i,j) = \forall x$ and $y \in \mathit{Term}$
$[M]\varphi \rightarrow [M_{(ij\leftarrow y)}]\{y/x\}\varphi$ (Universal Elimination)
where if $y \in \mathit{Var}$, then y is x or y is free for x in φ with y new in M

(S^S) $[MN]\varphi \rightarrow [M][N]\varphi$ (Split)

(J^S) For all $[M]$, $[N]$ such that $[M]$ is $\exists$-uniform or $[N]$ is $\forall$-uniform
$[M][N]\varphi \rightarrow [MN]\varphi$ (Join)

Rules:

Let $[M]\chi$ be a subformula of a formula ψ (possibly, $\psi = [M]\chi$). Fix an occurrence of $[M]\chi$ in ψ not in the scope of quantificational operators and call it $\tilde{\varphi}$. Let $\vec{x} = x_1, \ldots, x_n$ be a list of free variables of $[M]$ (not necessarily all). If $[N]$ is a uniform operator of ψ and $\tilde{\varphi}$ is in its scope, than no x_i can occur in $[N]$.

(*Rule of Universal Introduction*)
If $[M]$ is $\forall$-uniform, no variable x_i has free occurrences in subformulas of ψ that are in antecedent position (see page 4) with respect to $\tilde{\varphi}$, and $\tilde{\varphi}$ is in positive position in ψ, then

($\forall I^S$) $$\frac{\psi\{\tilde{\varphi}\}}{\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{y})}\}}$$ (Universal Introduction)

where for all i, either y_i is x_i or y_i new in $\tilde{\varphi}$.

(*Rule of Existential Elimination*)
If no variable x_i has free occurrences in subformulas of ψ that are in consequent position with respect to $\tilde{\varphi}$, and $\tilde{\varphi}$ is in negative position in ψ, then

($\exists E^S$) $$\frac{\psi\{\tilde{\varphi}\}}{\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\exists\vec{y})}\}}$$ (Existential Elimination)

where for all i, either y_i is x_i or y_i new in $\tilde{\varphi}$.

(*Rule of Modus Ponens*)

(MP) $$\frac{\varphi, \quad \varphi \rightarrow \psi}{\psi}$$ (Modus Ponens)

(*Rule of Necessitation*)
For $[M] \in OP$

(Nec) $$\frac{\vdash \varphi}{\vdash [M]\varphi}$$ (Necessitation)

As a remark to the deductive system for $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$, let us give a few examples of the application of rules ($\forall I^S$) and ($\exists E^S$). These rules have been stated in general terms to comprise several important cases among which the followings ($A \in$ *vOP*, no variable in A occurs free in M)

$$\frac{[M]\varphi}{[M_{(ij\leftarrow\forall y)}]\{y/x\}\varphi}; \qquad \frac{\chi \to [M]\varphi}{\chi \to [M_{(ij\leftarrow\forall y)}]\{y/x\}\varphi};$$

$$\frac{\chi \to [A][M]\varphi}{\chi \to [A][M_{(ij\leftarrow\forall y)}]\{y/x\}\varphi}; \qquad \frac{[M]\varphi \to \chi}{[M_{(ij\leftarrow\exists y)}]\{y/x\}\varphi \to \chi};$$

$$\frac{(\chi \to [M]\varphi) \to \psi}{(\chi \to [M_{(ij\leftarrow\exists y)}]\{y/x\}\varphi) \to \psi}; \qquad \frac{(\chi \to [A][M]\varphi) \to \psi}{(\chi \to [A][M_{(ij\leftarrow\exists y)}]\{y/x\}\varphi) \to \psi}.$$

Other cases are considered in Propositions 38-40.

Finally, note that the restriction of $(\forall I^S)$ to $\forall$-uniform operators is necessary to block incorrect deductions. Here is an example of a wrong deduction that we want to avoid: $\dfrac{\left[\begin{smallmatrix}\exists x\\ z\end{smallmatrix}\right]\varphi}{\left[\begin{smallmatrix}\exists x\\ \forall z\end{smallmatrix}\right]\varphi}$. This deduction is unsound. The hypothesis reads "For each given z, there exists x such that $\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right]\varphi$", while the conclusion states "There exists x such that for all z, $\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right]\varphi$". Clearly, the latter is not a logical consequence of the first.

4.1.1. *Soundness*

THEOREM 35 (SOUNDNESS). — *The axioms of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ are valid in any multi-agent Kripke model. Also, the rules of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ preserve truth in these models.*

PROOF. — Among the axioms, the cases of (PL), (K^S), $(\exists I^S)$, $(\forall E^S)$ are quite simple. Here we present the case of the normality axiom (K^S) only. Fix a model and an initial state s. If for some environment $\Im$, formula $[M]\varphi$ is satisfied at s and $[M]\psi$ is not, then there exists $[A] \in$ *Inst*(M) and a state s' such that $(s, s') \in R([\![A]\!])$ and $\varphi, \neg\psi$ hold at s' for $\Im$. But $[A](\varphi \to \psi)$ is satisfied at s for $\Im$, i.e., $\varphi \to \psi$ holds for $\Im$ at all states reachable through $[A]$. Thus, $\neg\varphi$ must be is satisfied at s' as well. Finally, $\neg[A](\varphi \to \psi)$ at s for $\Im$ and so $[M]\varphi$ fails at s. Contradiction.

Regarding the remaining axioms, consider first (S^S). This axiom holds since it is a consequence of the observation that in first-order logic formula $\exists\vec{x}\,\vec{y}\,\forall\vec{z}\,\vec{v}\,\varphi(\vec{x},\vec{y},\vec{z},\vec{v})$ implies $\exists\vec{x}\,\forall\vec{z}\,\exists\vec{y}\,\forall\vec{v}\,\varphi(\vec{x},\vec{y},\vec{z},\vec{v})$. The Join axiom (J^S) holds because the restriction on its application ensures that the interpretation of the two subformulas is obtained by instantiating the variables in the same order (namely, first all the existentially quantified and later the universally quantified variables).

Among the rules, the novelty lies with $(\forall I^S)$ and $(\exists E^S)$. We show how the argument runs for the first. The proof is by induction on the complexity of ψ. Let $\psi = [M]\chi$ with $\psi\{\tilde{\varphi}\} = \tilde{\varphi}$. We need to show that if $[M]\chi$ is true, then $[M_{(\vec{x}\leftarrow\forall\vec{x})}]\chi$ is true as well. We proceed by contradiction. Assume that $[M_{(\vec{x}\leftarrow\forall\vec{x})}]\chi$ is not true and fix a state s of the structure and an environment for which this formula is false. Then, there exists a sequence of terms $\vec{a}$ such that $[M_{(\vec{x}\leftarrow\vec{a})}]\{\vec{a}/\vec{x}\}\chi$ is false at s. That is, $[M]\chi$ is false at s for $\Im^{\vec{a}}_{\vec{x}}$. Contradiction.
Inductive step: assume the rule holds for $\psi\{\tilde{\varphi}\}$.
Case (a): $[N]\psi\{\tilde{\varphi}\}$ for some uniform $[N]$. By assumption, none of the x_i's occurs in $[N]$. Assume $[N]\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{x})}\}$ is false at a state s, then it follows easily that for some

environment $[N]\psi\{\tilde{\varphi}\}$ is false as well.
The following cases are treated together
(b) $\neg\neg\psi\{\tilde{\varphi}\}$,
(c) $\neg(\psi\{\tilde{\varphi}\} \to \varrho)$,
(d) $\neg\psi\{\tilde{\varphi}\} \to \varrho$, and
(e) $(\psi\{\tilde{\varphi}\} \to \varrho_1) \to \varrho_2$.
The first is trivial. The second and third are reduced to (e) by rewriting negation using $\to$ and $\bot$. ($\bot$ was introduced at page 4, it is easy to see that $\mathcal{M}, s \not\models \bot$ for all structures $\mathcal{M}$ and all states s.) Thus, we concentrate on case (e). Note that no subformula can be in antecedent position with respect to $\tilde{\varphi}$ unless it occurs in ψ. Assume that $(\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{x})}\} \to \varrho_1) \to \varrho_2$ fails for some environment $\Im$. Then, for $\Im$ the antecedent $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{x})}\} \to \varrho_1$ holds and the consequent ϱ_2 fails.
Subcase (e_1): by the inductive step, if $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{x})}\}$ fails for $\Im$, then $\psi\{\tilde{\varphi}\}$ fails as well. Thus, $\psi\{\tilde{\varphi}\} \to \varrho_1$ holds, by which we conclude that $(\psi\{\tilde{\varphi}\} \to \varrho_1) \to \varrho_2$ fails as we needed to show.
Subcase (e_2): assume ϱ_1 is true for $\Im$. Then, $\psi\{\tilde{\varphi}\} \to \varrho_1$ holds, by which we conclude again that $(\psi\{\tilde{\varphi}\} \to \varrho_1) \to \varrho_2$ fails and we are done.
Finally, it remains to verify case (f): $\varrho \to \psi\{\tilde{\varphi}\}$.
Assume that $\tilde{\varphi}$ in $\varrho \to \psi\{\tilde{\varphi}\}$ satisfies the conditions required by the rule. Then, if $\varrho \to \psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{x})}\}$ fails for an environment $\Im$, this means that ϱ holds for $\Im$ while $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{x})}\}$ fails. By the inductive step, $\psi\{\tilde{\varphi}\}$ fails for $\Im$ as well. Thus, $\varrho \to \psi\{\tilde{\varphi}\}$ fails also, and we are done. ■

4.1.2. *Derived rules and theorems*

Analogously to standard modal logic, using (Nec), (K^S), and (MP), one can prove the following distribution rule

PROPOSITION 36 (DISTRIBUTION RULE). — *The following rule is derivable:*

$$(RD)\ \frac{\varphi \to \psi}{[M]\varphi \to [M]\psi} \quad \textit{where } [M] \textit{ is } \forall\textit{-uniform.}$$

Having (RD) and (K^S), one proves other equivalences that are analogous to standard results of modal logic.

PROPOSITION 37 (DISTRIBUTION OVER $\wedge$). — *Let $[M]$ be a $\forall$-uniform operator, then*

$$[M](\varphi \wedge \psi) \leftrightarrow [M]\varphi \wedge [M]\psi.$$

The rules listed below are used in the completeness proof.
The first follows from $(\forall I^S)$ and $(\exists E^S)$.

PROPOSITION 38 (MODAL UNIVERSAL INTRODUCTION RULE). — *Let $[A]$ be uniform and $[M]$ $\forall$-uniform with $M(i,j) = x \in Var$. Then the following rule is derivable:*

(R∀I) $\dfrac{[A](\psi \to [M]\varphi)}{[A](\psi \to [M_{(ij \leftarrow \forall x)}]\varphi)}$

where x not free in $[A]$ nor in ψ.

The following is proved by applying $(\exists I^S)$.

PROPOSITION 39 (MODAL EXISTENTIAL INTRODUCTION RULE). — *Let $[A]$ be uniform, $M(i,j) = x \in Var$ and $a \in ActId$. Then the following rule is derivable:*

(R∃I) $\dfrac{[A](\psi \to [M_{(ij \leftarrow a)}]\{a/x\}\varphi)}{[A](\psi \to [M_{(ij \leftarrow \exists x)}]\varphi)}.$

By rule $(\exists E^S)$, we obtain

PROPOSITION 40 (MODAL EXISTENTIAL ELIMINATION RULE). — *Let $[A]$ be uniform and $M(i,j) = x \in Var$. Then the following rule is derivable:*

(R∃E) $\dfrac{[A]((\chi \to [M]\varphi) \to \psi)}{[A]((\chi \to [M_{(ij \leftarrow \exists x)}]\varphi) \to \psi)}$

where x not free in ψ.

We conclude the description of the logic $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ with a result that is quite important for applications in multi-agent systems. It shows that the capabilities of a group of agents (that is, the power of a *coalition* to force some sentence to become true) do not interfere with the capabilities of other groups (disjoint from the first). This is shown by proving that if a group of agents can force a sentence to become true and another group of agents can force a different sentence to become true, then the two groups can make true the conjunction of these sentences by acting concurrently.

If we focus on existential entries, a modal operator of our language can be seen as describing a kind of group (or coalition) to which agent i belongs as long as formula $[M]\varphi$ (with φ modal-free) shows existential entries in row i of M. In a sense, an agent participates to the group continuously, occasionally or never depending on the number of existential entries in its corresponding row.

Construct $\uplus$ was introduced in Definition 20.

THEOREM 41. — *Assume M, N are complementary, then*

$$\vdash ([M]\varphi \wedge [N]\psi) \to [M \uplus N](\varphi \wedge \psi)$$

PROOF. — We limit our proof to one-column operators in a two agent system. More precisely, we show the following:
$\vdash (\begin{bmatrix} \forall x \\ \exists y \end{bmatrix} \varphi \wedge \begin{bmatrix} \exists x \\ \forall y \end{bmatrix} \psi) \to \begin{bmatrix} \exists x \\ \exists y \end{bmatrix} (\varphi \wedge \psi).$
From axiom $(\forall E^S)$, we have $\vdash \begin{bmatrix} \forall x \\ y \end{bmatrix} \varphi \to \begin{bmatrix} x \\ y \end{bmatrix} \varphi$ and, similarly,

$\vdash \begin{bmatrix} x \\ \forall y \end{bmatrix} \psi \to \begin{bmatrix} x \\ y \end{bmatrix} \psi$.

By Proposition 37 and (PL), $\vdash (\begin{bmatrix} \forall x \\ y \end{bmatrix} \varphi \wedge \begin{bmatrix} x \\ \forall y \end{bmatrix} \psi) \to \begin{bmatrix} x \\ y \end{bmatrix} (\varphi \wedge \psi)$.

Applying $(\exists I^S)$ twice, we get $\vdash (\begin{bmatrix} \forall x \\ y \end{bmatrix} \varphi \wedge \begin{bmatrix} x \\ \forall y \end{bmatrix} \psi) \to \begin{bmatrix} \exists x \\ \exists y \end{bmatrix} (\varphi \wedge \psi)$.

By propositional logic,

$\vdash \begin{bmatrix} \forall x \\ y \end{bmatrix} \varphi \to \left(\begin{bmatrix} x \\ \forall y \end{bmatrix} \psi \to \begin{bmatrix} \exists x \\ \exists y \end{bmatrix} (\varphi \wedge \psi) \right)$

to which we can apply $(\exists E^S)$ obtaining

$\vdash \begin{bmatrix} \forall x \\ \exists y \end{bmatrix} \varphi \to \left(\begin{bmatrix} x \\ \forall y \end{bmatrix} \psi \to \begin{bmatrix} \exists x \\ \exists y \end{bmatrix} (\varphi \wedge \psi) \right)$.

By propositional logic again,

$\vdash \begin{bmatrix} x \\ \forall y \end{bmatrix} \varphi \to \left(\begin{bmatrix} \forall x \\ \exists y \end{bmatrix} \psi \to \begin{bmatrix} \exists x \\ \exists y \end{bmatrix} (\varphi \wedge \psi) \right)$

and by $(\exists E^S)$ again,

$\vdash \begin{bmatrix} \exists x \\ \forall y \end{bmatrix} \varphi \to \left(\begin{bmatrix} \forall x \\ \exists y \end{bmatrix} \psi \to \begin{bmatrix} \exists x \\ \exists y \end{bmatrix} (\varphi \wedge \psi) \right)$.

The latter is equivalent to $\vdash (\begin{bmatrix} \forall x \\ \exists y \end{bmatrix} \varphi \wedge \begin{bmatrix} \exists x \\ \forall y \end{bmatrix} \psi) \to \begin{bmatrix} \exists x \\ \exists y \end{bmatrix} (\varphi \wedge \psi)$ and we are done. ∎

4.2. *Completeness*

In this section, we prove that the logic $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ (for $k \geq 1$) is complete for the class of multi-agent Kripke frames. Our proof of completeness follows the Henkin's method generalized to first-order modal logics. As usual, below we assume that an index k has been fixed.

The standard notion of ω-*completeness* needs to be adapted to our language. In $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$, one cannot express the existential quantifier in terms of the universal one. As a consequence, ω-completeness in our logic must consider both positive and negative expressions. In addition, this notion has to take into account the special interplay among quantificational operators and classical negation. For these reasons, our definition of ω-completeness splits in two cases.

DEFINITION 42 (ω-COMPLETE SET). —

A set $\Sigma \subseteq \mathcal{L}_{\mathcal{MA}}$ is ω-complete *if it satisfies the following conditions:*

(i_ω) *Let $[M]$ be $\forall$-uniform with universally quantified variables $\vec{x}$. Let $[M]\chi$ be a subformula of ψ, and $\tilde{\varphi}$ a positive occurrence of $[M]\chi$ in ψ, $\tilde{\varphi}$ not in the scope of a quantificational operator,*
if $\Sigma \vdash \psi\{\tilde{\varphi}_{(\vec{x} \leftarrow \vec{a})}\}$ for all sequences of constants $\vec{a}$ (of the right length), then $\Sigma \vdash \psi$;

(ii_ω) *Let* $[M]$ *be a quantificational operator with some existentially quantified variables* $\vec{x}$*. Let* $[M]\chi$ *be a subformula of* ψ *and* $\tilde{\varphi}$ *a negative occurrence of* $[M]\chi$ *in* ψ*,* $\tilde{\varphi}$ *not in the scope of a quantificational operator,*
if $\Sigma \vdash \psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\vec{a})}\}$ *for all sequences of constants* $\vec{a}$ *(of the right length), then* $\Sigma \vdash \psi$*.*

DEFINITION 43 (MAXIMAL, CONSISTENT, AND SATURATED SETS). —

(i) A set Σ *is* maximal *if either* $\varphi \in \Sigma$ *or* $\neg\varphi \in \Sigma$ *for every formula* φ *in the language.*

(ii) A set of formulas Σ *is* consistent *if* $\Sigma \nvdash \bot$*.*

(iii) A set Σ *is* saturated *if it is maximal, consistent, and* ω*-complete.*

LEMMA 44. — *Fix* φ *with no free occurrence of* y*,* y *new in* M*, and* y *free for* x *in* φ*.*

If $M(i,j) = \forall x$*, then*

$$\vdash [M]\varphi \leftrightarrow [M_{(ij\leftarrow\forall y)}]\{y/x\}\varphi$$

If $M(i,j) = \exists x$*, then*

$$\vdash [M]\varphi \leftrightarrow [M_{(ij\leftarrow\exists y)}]\{y/x\}\varphi$$

PROOF. — The first equivalence requires some work because of the restriction in ($\forall I^S$). For the sake of simplicity, we prove it for the simple modality $[M] = \left[\begin{smallmatrix}\forall x\\ \exists z\end{smallmatrix}\right]$, thus assuming $k = 2$.

By $(\forall E^S), \vdash \begin{bmatrix}\forall x\\ z\end{bmatrix}\varphi \rightarrow \begin{bmatrix}y\\ z\end{bmatrix}\{y/x\}\varphi$ (since y free for x in φ)

By $(\forall I^S), \vdash \begin{bmatrix}\forall x\\ z\end{bmatrix}\varphi \rightarrow \begin{bmatrix}\forall y\\ z\end{bmatrix}\{y/x\}\varphi$

By $(\exists I^S), \vdash \begin{bmatrix}\forall y\\ z\end{bmatrix}\{y/x\}\varphi \rightarrow \begin{bmatrix}\forall y\\ \exists z\end{bmatrix}\{y/x\}\varphi$

By $(MP), \vdash \begin{bmatrix}\forall x\\ z\end{bmatrix}\varphi \rightarrow \begin{bmatrix}\forall y\\ \exists z\end{bmatrix}\{y/x\}\varphi$

By $(\exists E^S), \vdash \begin{bmatrix}\forall x\\ \exists z\end{bmatrix}\varphi \rightarrow \begin{bmatrix}\forall y\\ \exists z\end{bmatrix}\{y/x\}\varphi$

The other direction holds as well since the conditions imply x is free for y in $\{y/x\}\varphi$.

For the second equivalence. $\vdash [M_{(ij\leftarrow x)}]\varphi \rightarrow [M_{(ij\leftarrow\exists y)}]\{y/x\}\varphi$ is an instance of ($\exists I^S$) and, by ($\exists E^S$), $\vdash [M_{(ij\leftarrow\exists x)}]\varphi \rightarrow [M_{(ij\leftarrow\exists y)}]\{y/x\}\varphi$ where $[M_{(ij\leftarrow\exists x)}]$ is $[M]$. The other direction is proved similarly. ■

Fix sets *ActId*, *PropId*, and *Var*, and let $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ be the language thus obtained. Let Σ be an arbitrary consistent set of formulas of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$. We prove completeness by showing that Σ has a model. The first step consists in showing how to extend Σ to a saturated set applying a version of the Henkin method [GAR 01].

Let *Cn* be a denumerable set of action identifiers with *Cn* $\cap$ *ActId* $= \emptyset$. Let $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$ be the language $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ with action identifiers from $ActId^+ = ActId \cup Cn$. From now on, we will refer to elements of $ActId^+$ indifferently as action identifiers or *constants*. Note that if Σ is consistent in $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$, then it is consistent in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$ as well. From now on, we will work in this extended system.

Let $\tau(1), \tau(2), \ldots$ be a list of all $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$-formulas such that if a formula contains n (not nested) quantificational operators, then it occurs n times in the list τ; the first time associated with the first not nested quantificational operator (from left to right), the second time associated with the second not nested quantificational operator and so on till the n-th occurrence of the formula in τ. If ψ is a formula with quantificational operators and $\psi = \tau(h)$, then the *label of* ψ at h is the occurrence of the quantificational operator with which the formula is associated at h. Thus, if $M \in$ *qOP*, then formula $\psi = p_0 \rightarrow [M]p_1$ occurs once in τ with label the only occurrence of M in ψ. Instead, formula $[M]p_0 \rightarrow [M][M]p_1$ occurs twice. It occurs once with label the first occurrence (from left to right) of M, once with label the second occurrence of M, and it does not occur with label the third occurrence (since the latter is in the scope of another quantificational operator, thus nested).

We do not mark labels explicitly in our notation since the label of $\psi = \tau(h)$ is clear from the formula ψ and its occurrences in $\tau(1), \ldots, \tau(h-1)$, if any. A formula without quantificational operators has no label. Given a formula ψ and its label M, if $\psi = \tau(h)$, we write $\tilde{\varphi}(h)$ for the *unique occurrence* of the subformula $[M]\chi$ where M is the label of ψ and χ its scope in ψ.

To obtain a saturated set from Σ, we proceed by recursion constructing a sequence of consistent sets Σ_h in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$ such that $\Sigma_0 =_{def} \Sigma$ and $\Sigma_h \subseteq \Sigma_{h+1}$. (We write $\Sigma \cup \varphi$ for the set $\Sigma \cup \{\varphi\}$.)

Suppose Σ_h has been constructed.

(Case A) Let $\tau(h+1) = \psi$ be a formula with label a quantificational $\forall$-uniform operator $[M]$ and let $\vec{y}$ be the universally quantified variables of $[M]$. Let $\tilde{\varphi}(h+1)$ be negative in ψ, then
if $\Sigma_h \cup \tau(h+1)$ is consistent, put $\Sigma_{h+1} =_{def} \Sigma_h \cup \tau(h+1) \cup \psi\{\tilde{\varphi}(h+1)_{(\vec{y} \leftarrow \vec{a})}\}$ where
$\vec{a}$ is a sequence (of the right length) of constants of *Cn* new in $\Sigma_h \cup \tau(h+1)$;[11]
if $\Sigma_h \cup \tau(h+1)$ is not consistent, put $\Sigma_{h+1} =_{def} \Sigma_h$.

(Case B) Let $\tau(h+1) = \psi$ be a formula with label a quantificational operator $[M]$ with one or more existential entries, and let $\vec{x}$ be its existentially quantified variables (M

11. We insist that the substitution is applied to the formula occurrence $\tilde{\varphi}(h+1)$ only.

may contain universally quantified variables as well). If $\tilde{\varphi}(h+1)$ is positive in ψ, then
if $\Sigma_h \cup \tau(h+1)$ is consistent, put $\Sigma_{h+1} =_{def} \Sigma_h \cup \tau(h+1) \cup \psi\{\tilde{\varphi}(h+1)_{(\vec{x}\leftarrow\vec{a})}\}$ where
$\vec{a}$ is a sequence (of the right length) of constants of *Cn* new in $\Sigma_h \cup \tau(h+1)$;[12]
if $\Sigma_h \cup \tau(h+1)$ is not consistent, put $\Sigma_{h+1} =_{def} \Sigma_h$.

(Case C) If $\tau(h+1)$ falls in none of the above cases, then
if $\Sigma_h \cup \{\tau(h+1)\}$ is consistent, put $\Sigma_{h+1} =_{def} \Sigma_h \cup \tau(h+1)$.
if $\Sigma_h \cup \{\tau(h+1)\}$ is not consistent, put $\Sigma_{h+1} =_{def} \Sigma_h$.

We need to show that Σ_{h+1} is consistent if Σ_h is.

(Case A) Suppose $\tau(h+1) = \psi$, $\Sigma_h \cup \tau(h+1)$ is consistent, and Σ_{h+1} is not. From the construction, $\Sigma_h \cup \tau(h+1) \vdash \neg\psi\{\tilde{\varphi}(h+1)_{(\vec{y}\leftarrow\vec{a})}\}$.
Recall that the elements of $\vec{a}$ are new in $\Sigma_h \cup \tau(h+1)$.
Since a proof of $\neg\psi\{\tilde{\varphi}(h+1)_{(\vec{y}\leftarrow\vec{a})}\}$ uses only finitely many formulas, let $\vec{x}$ be variables not occurring in the proof and substitute $\vec{x}$ for $\vec{a}$ in the whole proof. Formula $\tilde{\varphi}(h+1)_{(\vec{y}\leftarrow\vec{x})}$ is positive in $\neg\psi\{\tilde{\varphi}(h+1)_{(\vec{y}\leftarrow\vec{x})}\}$ and the latter satisfies the conditions in $(\forall I^S)$. By applying $(\forall I^S)$ over the x_i's (and Lemma 44), one obtains $\Sigma_h \cup \tau(h+1) \vdash \neg\psi$, contradicting the assumption that $\Sigma_h \cup \tau(h+1)$ is consistent.

(Case B) Suppose $\tau(h+1) = \psi$, $\Sigma_h \cup \tau(h+1)$ is consistent, and Σ_{h+1} is not. From the construction, $\Sigma_h \cup \tau(h+1) \vdash \neg\psi\{\tilde{\varphi}(h+1)_{(\vec{x}\leftarrow\vec{a})}\}$. In $\neg\psi\{\tilde{\varphi}(h+1)_{(\vec{x}\leftarrow\vec{a})}\}$, the occurrence $\tilde{\varphi}(h+1)_{(\vec{x}\leftarrow\vec{a})}$ is negative. Arguing as in the previous case, we can apply $(\exists E^S)$ since the conditions are satisfied. Thus, by Lemma 44, one obtains $\Sigma_h \cup \tau(h+1) \vdash \neg\psi$ contradicting the assumption that $\Sigma_h \cup \tau(h+1)$ is consistent.

(Case C) By construction.

Finally, put $\Sigma_\infty = \bigcup_{i=1}^{\infty} \Sigma_i$.

It turns out that Σ_∞ is maximal, consistent and ω-complete in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$. Consistency follows from the consistency of Σ_i for each i. Also, the process guarantees Σ_∞ is maximal since only formulas not consistent with some subset Σ_i are left out in the construction of Σ_∞. Regarding ω-completeness, we need an argument.

Consider formulas ψ and $\neg\psi$ with quantificational operators. Without loss of generality, let $\tau(r) = \neg\psi$ be the first occurrence of $\neg\psi$ in τ. As usual, $\tilde{\varphi}(r)$ is the label of $\neg\psi$ at r. We write $\tilde{\varphi}$ for the first occurrence of formula $\tilde{\varphi}$ in ψ. (Informally, $\tilde{\varphi}$ in ψ "corresponds" to the occurrence of $\tilde{\varphi}(r)$ in $\neg\psi$.) Note that, if $\tilde{\varphi}(r)$ is positive in $\neg\psi$, then $\tilde{\varphi}$ is negative in ψ, and vice versa.

(i_ω) Assume $[M]$, $\vec{x}$, ψ, $\tilde{\varphi}$ satisfy the conditions in (i_ω), and let $\Sigma_\infty \vdash \psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\vec{a})}\}$ for all sequences $\vec{a}$ (of the right length). If $\Sigma_r \cup \tau(r)$ is consistent, then by (Case A) of the construction there exists a sequence $\vec{a}$ such that $\Sigma_\infty \vdash (\neg\psi)\{\tilde{\varphi}(r)_{(\vec{x}\leftarrow\vec{a})}\}$. But $(\neg\psi)\{\tilde{\varphi}(r)_{(\vec{x}\leftarrow\vec{a})}\}$ and $\neg(\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\vec{a})}\})$ are the same formula. Contradiction. Thus, $\Sigma_\infty \nvdash \neg\psi$ and, by maximality, $\Sigma_\infty \vdash \psi$.

12. As before, note that the substitution is applied to the formula occurrence $\tilde{\varphi}(h+1)$ only.

(ii_ω) Assume $[M]$, $\vec{x}$, ψ, $\tilde{\varphi}$ satisfy the conditions in (ii_ω), and let $\Sigma_\infty \vdash \psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\vec{a})}\}$ for all sequences $\vec{a}$ (of the right length). If $\Sigma_r \cup \tau(r)$ is consistent, then by (Case B) of the construction there exists a sequence $\vec{a}$ such that $\Sigma_\infty \vdash (\neg\psi)\{\tilde{\varphi}(r)_{(\vec{x}\leftarrow\vec{a})}\}$. As before, this leads to a contradiction. Thus, $\Sigma_\infty \nvdash \neg\psi$ and, by maximality, $\Sigma_\infty \vdash \psi$.

Since we can apply the same argument for all other labels of ψ, we conclude that Σ_∞ is ω-complete.

We now state this result officially.

LEMMA 45. — *Let Σ be a consistent set in $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$. We can find a saturated set of $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$ which is an extension of Σ.*

DEFINITION 46 (CANONICAL FRAME). —

The canonical frame $\mathcal{K}_\mathcal{C}$ *for $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$ is frame $\langle W_\mathcal{C}, Act_\mathcal{C}; R_\mathcal{C}\rangle$ where:*

- $W_\mathcal{C}$ is the set of all saturated sets in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$;

- $Act_\mathcal{C}$ is the set of constants in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$, i.e. $Act_\mathcal{C} = ActId^+$;

- For $s, s' \in W_\mathcal{C}$ and a k-action $\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$ for $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$, the relation $R_\mathcal{C}$ is defined by the following condition:

$$(s, s') \in R_\mathcal{C} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix} \text{ if and only if } \left\{ \varphi \mid \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{bmatrix} \varphi \in s \right\} \subset s'.$$

DEFINITION 47 (CANONICAL MODEL). —

The canonical model $\mathcal{M}_\mathcal{C}$ *for $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$ is obtained by augmenting the canonical frame $\mathcal{K}_\mathcal{C}$ with a valuation function $[\![\cdot]\!]_\mathcal{C}$ where $[\![\cdot]\!]_\mathcal{C}$ is defined on $\varphi \in PropId$ by $[\![\varphi]\!]_\mathcal{C} = \{s \mid \varphi \in s\}$ and it is defined to be the identity function on* ActId$^+$.

We write $\langle W_\mathcal{C}, Act_\mathcal{C}; R_\mathcal{C}; [\![\cdot]\!]_\mathcal{C}\rangle$ or, indifferently, $\langle \mathcal{K}_\mathcal{C}, [\![\cdot]\!]_\mathcal{C}\rangle$ for the canonical model.

By construction, any k-action A is also an operator in *cOP* since $A = [\![A]\!]_\mathcal{C}$ in the canonical model. We will take advantage of this double role of k-actions and talk informally of modal operators as instances of other operators.

Note that a saturated set s enjoys crucial properties like:

(i) $\varphi \notin s$ if and only if $\neg\varphi \in s$;

(ii) if $\varphi \to \psi \in s$ and $\varphi \in s$, then $\psi \in s$;

(iii) if $\varphi \to \psi$, $\neg\psi$ both in s, then $\neg\varphi \in s$;

(iv) if $s \vdash \varphi$, then $\varphi \in s$;

(v) if $(\forall I^S)$ applies to $\psi\{\tilde{\varphi}\} \in s$, then $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\forall\vec{y})}\} \in s$;

(vi) if $(\exists E^S)$ applies to $\psi\{\tilde{\varphi}\} \in s$, then $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\exists\vec{y})}\} \in s$;

(vii) Let $[A]$ be uniform and $[M]$ be $\forall$-uniform with (some or all) variables in $\vec{x}$: if $s \vdash [A](\psi \to [M_{(\vec{x}\leftarrow\vec{b})}]\{\vec{b}/\vec{x}\}\varphi)$ holds for any sequence $\vec{b}$ of constants, then $s \vdash [A](\psi \to [M]\varphi)$;

(viii) Let $[A]$ be uniform, $M(i_1, j_1) = \exists x_1, \ldots, M(i_p, j_p) = \exists x_p$ be all the existential entries of $[M]$ and $\vec{y}$ some or all the universally quantified variables of M. Then, if there exist constants $a_1, \ldots, a_p$, such that $s \vdash [A]\neg(\psi \to [M_{(\vec{x},\vec{y}\leftarrow\vec{a},\vec{b})}]\{\vec{a}, \vec{b}/\vec{x}, \vec{y}\}\varphi)$ for all sequences $\vec{b}$ of constants, then $s \vdash [A]\neg(\psi \to [M]\varphi)$.

Properties (i)-(vi) hold true since s is maximal and consistent; the remaining two are special applications of ω-completeness.

THEOREM 48. — *Let $\Im$ be an environment and $[\![\cdot]\!]_{\mathcal{C}}$ the valuation function extended over vOP as in Definition 22. For all uniform operators* $[A], [B]$

$$R_{\mathcal{C}}([\![AB]\!]_{\mathcal{C}}) = R_{\mathcal{C}}([\![A]\!]_{\mathcal{C}}) \circ R_{\mathcal{C}}([\![B]\!]_{\mathcal{C}})$$

PROOF. — Without loss of generality, we assume $A, B \in$ *cOP*.

$\supseteq$) This inclusion follows by induction from axioms (S^S) and (J^S), and from the definition of the frame relation $R_{\mathcal{C}}$ over k-actions.

$\subseteq$) Assume $(s, s') \in R_{\mathcal{C}}([\![AB]\!]_{\mathcal{C}})$, that is, $[AB]\chi \in s$ implies $\chi \in s'$.
Let $D = \{\varphi \mid [A]\varphi \in s\}$ and $\Delta = \{\varphi \mid [A]\varphi \in s\} \cup \{\neg[B]\psi \mid \psi \notin s'\}$.

We begin by proving that Δ is consistent and that subset D is ω-complete. Then, we show that D can be extended to a saturated set containing Δ. In this way we isolate a saturated set s'' in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$ such that $(s, s'') \in R_{\mathcal{C}}([\![A]\!]_{\mathcal{C}})$ and $(s'', s') \in R_{\mathcal{C}}([\![B]\!]_{\mathcal{C}})$. Once this is done, by definition of relation $\circ$, we conclude that $(s, s') \in R_{\mathcal{C}}([\![AB]\!]_{\mathcal{C}})$ implies $(s, s') \in R_{\mathcal{C}}([\![A]\!]_{\mathcal{C}}) \circ R_{\mathcal{C}}([\![B]\!]_{\mathcal{C}})$.

CLAIM 49. — Δ is consistent.

Suppose that Δ is not consistent. Choose $\{\varphi_1, \ldots, \varphi_p\} \subseteq \Delta$ $(p \geq 0)$ such that $[A]\varphi_i \in s$ and $\{\neg[B]\psi_1, \ldots, \neg[B]\psi_q\} \subseteq \Delta$ $(q \geq 0)$ such that $\psi_i \notin s'$ with $\{\varphi_1, \ldots, \varphi_p, \neg[B]\psi_1, \ldots, \neg[B]\psi_q\}$ not consistent. Put $\psi = \psi_1 \vee \ldots \vee \psi_q$. Using (K^S), $[B]\psi_1 \wedge \ldots \wedge [B]\psi_q \to [B]\psi$. By (MP) and (PL), we have $\varphi_1 \wedge \ldots \wedge \varphi_p \to [B]\psi$ from which, by (Nec) and (K^S) again, $[A]\varphi_1 \wedge \ldots \wedge [A]\varphi_p \to [A][B]\psi$. From $[A], [B] \notin$ *qOP*, (S^S), and (J^S), one shows $[A][B]\psi \leftrightarrow [AB]\psi$. This implies $\psi \in s'$ so that, by maximality, at least one among $\psi_1, \ldots, \psi_q$ is in s'. Contradiction. Thus, Δ is consistent. ■(Claim 49)

The crucial step is to find an extension of Δ which is not only consistent but ω-complete as well. We build such an extension of Δ starting from set $D = \{\varphi \mid [A]\varphi \in s\}$ which is already ω-complete as the following argument shows.

CLAIM 50. — D is ω-complete.

Assume ψ, $[M]$, $\vec{x}$, $\tilde{\varphi}$ satisfy the conditions of (i_ω). If $\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\vec{a})}\}$ is in D, then $[A]\psi\{\tilde{\varphi}_{(\vec{x}\leftarrow\vec{a})}\} \in s$. This happens for all sequences $\vec{a}$ (of the right length). By ω-completeness of s, we obtain $[A]\psi\{\tilde{\varphi}\} \in s$ so that, by definition of D, $\psi\{\tilde{\varphi}\} \in D$. Analogously for (ii_ω). ■(Claim 50)

Let $F = \{\neg[B]\psi \mid \psi \notin s'\}$ so that $\Delta = D \cup F$.

We now show how to extend D to a saturated set s'' containing F.

Fix a list $\sigma(1), \sigma(2), \ldots$ of all formulas in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$ (differently from the list τ considered earlier, in σ each formula occurs only once). Put $E_0 = D$, $\sigma(h+1) = \psi$, and define E_{h+1} by cases as follows:

(i) Let ψ be a formula with some negative $\forall$-uniform operators (that is, with some $\forall$-uniform operators in negative position) not in the scope of other quantificational operators or some positive $\exists$-uniform operators not in the scope of other quantificational operators (both conditions might hold).
For the sake of clarity, we present this case through an example. Assume that

a) there are two negative occurrences of $\forall$-uniform operators, namely $[M_1]$ and $[M_2]$ (possibly different occurrences of the same operator) with quantified variables $\vec{x}_1$ and $\vec{x}_2$, respectively, and

b) there are two positive occurrences of $\exists$-uniform operators, namely $[N_1]$ and $[N_2]$ (possibly different occurrences of the same operator) with quantified variables $\vec{y}_1$ and $\vec{y}_2$, respectively;

c) there are no other quantificational operators in ψ satisfying the same conditions.

Let $\tilde{\varphi}(M)$ be the smallest subformula of ψ with operator $M \in \{M_1, M_2, N_1, N_2\}$.

If there exist sequences of terms $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$ such that
$E_h \cup F \cup \psi \cup \psi\{\tilde{\varphi}(M_1)_{(\vec{x}_1\leftarrow\vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2\leftarrow\vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1\leftarrow\vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2\leftarrow\vec{b}_2)}\}$
is consistent, then put $E_{h+1} =_{def}$
$E_h \cup \psi \cup \psi\{\tilde{\varphi}(M_1)_{(\vec{x}_1\leftarrow\vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2\leftarrow\vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1\leftarrow\vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2\leftarrow\vec{b}_2)}\}$.
Put $E_{h+1} =_{def} E_h$, otherwise.

(ii) If ψ does not fall in the previous case and $E_h \cup F \cup \psi$ is not consistent, then put $E_{h+1} =_{def} E_h$.

(iii) If none of the above applies, put $E_{h+1} =_{def} E_h \cup \psi$.

CLAIM 51. — E_h is consistent and ω-complete for all h.

Since $E_0 = D \subseteq \Delta$ and Δ is consistent (Claim 49), the construction above guarantees that E_h is consistent for every h. For the other property we proceed by induction on h. We know from Claim 50 that $E_0 = D$ is ω-complete. Suppose that E_h is ω-complete. We show E_{h+1} is ω-complete as well. Let $E_{h+1} = E_h \cup \psi$ (for E_{h+1} obtained as in case (i) it suffices to apply the argument twice augmenting E_h with one formula at a time).

Property (i_ω).
Let $E_h \cup \psi \vdash \chi\{\tilde{\varphi}(h+1)_{(\vec{y}\leftarrow\vec{a})}\}$ for all sequences $\vec{a}$ (of the right length), where $\tilde{\varphi}(h+1)$ is a positive occurrence in χ of a $\forall$-uniform operator with quantified variables $\vec{y}$. Then, $E_h \vdash \psi \to \chi\{\tilde{\varphi}(h+1)_{(\vec{y}\leftarrow\vec{a})}\}$ for all $\vec{a}$. Note that $\tilde{\varphi}(h+1)$ is positive in $\psi \to \chi$. By ω-completeness of E_h, we obtain $E_h \vdash \psi \to \chi$ and, thus, $E_h \cup \psi \vdash \chi$.

Property (ii_ω).
Let $E_h \cup \psi \vdash \chi\{\tilde{\varphi}(h+1)_{(\vec{x}\leftarrow\vec{a})}\}$ for all sequences $\vec{a}$ (of the right length), where $\tilde{\varphi}(h+1)$ is a negative occurrence of an operator with (one or more) existential entries. Let $\vec{x}$ be all its existentially quantified variables. Then, $E_h \vdash \psi \to \chi\{\tilde{\varphi}(h+1)_{(\vec{x}\leftarrow\vec{a})}\}$ for all $\vec{a}$. As before, we obtain $E_h \vdash \psi \to \chi$ and, thus, $E_h \cup \psi \vdash \chi$. ■ (Claim 51)

Finally, put $s'' = \bigcup E_h$. We need to show:

(I) $F \subset s''$ and

(II) s'' is saturated.

(I) Let $\sigma(h+1) = \neg[B]\chi \in F$. Thus, $E_h \cup F \cup \sigma(h+1) = E_h \cup F$ which is consistent by construction. It suffices to show $\sigma(h+1) \in E_{h+1}$. From the construction, cases (ii) and (iii) are trivial. For case (i), without loss of generality, assume that formula $\neg[B]\chi$ has operator occurrences $\{M_1, M_2, N_1, N_2\}$ as described in (i) for $\psi = \neg[B]\chi$. As before, let $\tilde{\varphi}(M)$ be the smallest subformula of $\neg[B]\chi$ containing operator $M \in \{M_1, M_2, N_1, N_2\}$.

We need to show that there are sequences $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$ such that $E_{h+1} = E_h \cup \neg[B]\chi \cup \neg[B]\chi\{\tilde{\varphi}(M_1)_{(\vec{x}_1\leftarrow\vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2\leftarrow\vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1\leftarrow\vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2\leftarrow\vec{b}_2)}\}$.

Since $\neg[B]\chi \in F$, then $\neg\chi \in s'$. Thus, $\chi \notin s'$. By ω-completeness, $\neg\chi\{\tilde{\varphi}(M_1)_{(\vec{x}_1\leftarrow\vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2\leftarrow\vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1\leftarrow\vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2\leftarrow\vec{b}_2)}\} \in s'$ for some $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$.
Let $\gamma = \neg[B]\chi\{\tilde{\varphi}(M_1)_{(\vec{x}_1\leftarrow\vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2\leftarrow\vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1\leftarrow\vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2\leftarrow\vec{b}_2)}\}$. Clearly, $\gamma \in F$ thus $E_h \cup F \cup \neg[B]\chi \cup \gamma = E_h \cup F$ which is consistent. According to the construction rules, we have $\neg[B]\chi \in E_{h+1}$. We conclude that $F \subset s''$.

(II) From Claim 51 it follows that s'' is consistent and ω-complete. It remains to show that s'' is maximal.

Suppose not. Let $\psi \notin s''$ and assume $s'' \cup \psi$ is consistent . Since σ lists all formulas in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$, for some index h we have $\sigma(h+1) = \psi$. Then, $E_h = E_{h+1}$. We check if the three cases in the construction are compatible with this result:

Case (i). As before, we check this case through an example. Assume ψ contains exactly two negative occurrences of $\forall$-uniform operators (not in the scope of other quantificational operators), namely $[M_1]$ and $[M_2]$ with quantified variables $\vec{x}_1$ and $\vec{x}_2$, respectively; and exactly two positive occurrences of $\exists$-uniform operators (not in the scope of other quantificational operators), namely $[N_1]$ and $[N_2]$ with quantified variables $\vec{y}_1$ and $\vec{y}_2$, respectively. Let $\tilde{\varphi}(M)$ be the subformula of ψ corresponding to label $M \in \{M_1, M_2, N_1, N_2\}$.

Since ψ has not been included in E_{h+1}, it must be that
$E_h \cup F \cup \psi \vdash \neg\psi\{\tilde{\varphi}(M_1)_{(\vec{x}_1 \leftarrow \vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2 \leftarrow \vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1 \leftarrow \vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2 \leftarrow \vec{b}_2)}\}$ for all sequences $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$, i.e.,
$E_h \cup F \vdash \psi \rightarrow \neg\psi\{\tilde{\varphi}(M_1)_{(\vec{x}_1 \leftarrow \vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2 \leftarrow \vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1 \leftarrow \vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2 \leftarrow \vec{b}_2)}\}$ for all sequences $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$. Since $F \subset s''$, then
$s'' \vdash \psi \rightarrow \neg\psi\{\tilde{\varphi}(M_1)_{(\vec{x}_1 \leftarrow \vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2 \leftarrow \vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1 \leftarrow \vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2 \leftarrow \vec{b}_2)}\}$.
But s'' is ω-complete so that, by property (i_ω) on $\tilde{\varphi}(M_1), \tilde{\varphi}(M_2)$ and property (ii_ω) on $\tilde{\varphi}(N_1), \tilde{\varphi}(N_2)$ (plus Lemma 44), one gets $s'' \vdash \psi \rightarrow \neg\psi$. Thus, $s'' \cup \psi$ is not consistent. Contradiction.

Case (ii): The consistency of $s'' \cup \psi$ would contradict $E_h = E_{h+1}$.

Case (iii): We get a contradiction from $\psi \notin s''$. ■

From section 3.2, the canonical model $\mathcal{M}_\mathcal{C}$ is a Kripke structure for multi-agent systems.

Note that, while proving Theorem 48, we have applied an argument that yields another important property. We isolate it here.

THEOREM 52. — *Let s be a saturated set of formulas and consider $\Gamma = \{\psi \mid [A]\psi \in s\}$. Then,*

$$\Gamma \vdash \chi \iff \chi \in s' \textit{ for all } s' \textit{ saturated with } \Gamma \subseteq s'$$

PROOF. — (Left to Right) Suppose that $\Gamma \vdash \chi$. Then, there are $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\varphi_1, \ldots, \varphi_n \rightarrow \chi$ in $(\mathcal{L}^+)^{\mathcal{S}}_{\mathcal{MA}(k)}$. The very same proof then holds in all s' containing Γ and, by the properties of saturated sets (see page 38), $\chi \in s'$.

(Right to Left) Assume not, then $\Gamma \cup \neg\chi$ is consistent . From Claim 50 of Theorem 48 we conclude that Γ is ω-complete as well. It remains to show that the consistent and ω-complete set Γ can be extended to a saturated set s'' (in the same language) containing formula $\neg\chi$. This result is obtained following the construction of s'' in the proof of Theorem 48 where we substitute Γ for Δ and $\{\neg\chi\}$ for F. What changes is the argument to prove $\neg\chi \in s''$, i.e., case (I) in the theorem above.

Let $F = \{\neg\chi\}$ and suppose s'' has been constructed following the work done in the proof of the previous theorem. In particular, $E_h \cup F$ is consistent for all h. Let $\sigma(h+1) = \neg\chi$. We show $\neg\chi \in E_{h+1}$. Suppose not. Case (ii) does not apply and case (iii) gives immediately $\sigma(h+1) \in E_{h+1}$. Suppose now that $\neg\chi$ falls under case (i). If $\neg\chi \notin E_{h+1}$, then for $\neg\chi = \psi$ we have
$E_h \cup F \vdash \neg\psi\{\tilde{\varphi}(M_1)_{(\vec{x}_1 \leftarrow \vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2 \leftarrow \vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1 \leftarrow \vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2 \leftarrow \vec{b}_2)}\}$ for all sequences $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$, i.e.,
$E_h \vdash \psi \rightarrow \neg\psi\{\tilde{\varphi}(M_1)_{(\vec{x}_1 \leftarrow \vec{a}_1)}, \tilde{\varphi}(M_2)_{(\vec{x}_2 \leftarrow \vec{a}_2)}, \tilde{\varphi}(N_1)_{(\vec{y}_1 \leftarrow \vec{b}_1)}, \tilde{\varphi}(N_2)_{(\vec{y}_2 \leftarrow \vec{b}_2)}\}$.
By ω-completeness of E_h, we get $E_h \vdash \psi \rightarrow \neg\psi$, i.e., $E_h \vdash \neg\chi \rightarrow \chi$ or $E_h \cup F \vdash \chi$, contradicting the consistency of $E_h \cup F$. Finally, $\neg\chi \in E_{h+1}$. ■

We pause for a moment to verify an important property of the system.

PROPOSITION 53. — *Let $\Im$ be an environment and $[M]$ an operator with $\vec{x}$ all its existentially quantified variables (possibly none), and $\vec{y}$ all its universally quantified variables (possibly none). Fix a state s in the canonical model $\mathcal{M}_\mathcal{C}$ and a formula*

$$\neg[M]\varphi \in s$$

Then, for any sequence $\vec{a}$ of constants there exist a sequence $\vec{b}$ and a state s' such that

$$(s, s') \in R_\mathcal{C}([\![M_{(\vec{x},\vec{y}\leftarrow\vec{a},\vec{b})}]\!]_\mathcal{C}) \quad \textit{and} \quad \neg\{\vec{a},\vec{b}/\vec{x},\vec{y}\}\varphi \in s'$$

PROOF. — (Without loss of generality, let us assume $[M]$ has no parameter entry.)

Consider first a simple case. Let $[M] \in$ *cOP* and $\Sigma = \{\psi \mid [M]\psi \in s\} \cup \{\neg\varphi\}$. By (the proofs of) Theorem 48 and by Theorem 52, Σ is consistent and $\{\psi \mid [M]\psi \in s\}$ is ω-complete. Following the proof of Theorem 52, one obtains a saturated extension s' (thus a state in the canonical model) such that $\Sigma \subseteq s'$ and, thus, $\neg\varphi \in s'$. At this point $(s, s') \in R_\mathcal{C}([\![M]\!]_\mathcal{C})$ follows from $\Sigma \subseteq s'$ and the definition of $R_\mathcal{C}$.

For the general case, let $[M] \in$ *qOP*. From the contrapositive of $(\exists I^S)$ on $\neg[M]\varphi$ one gets $\neg[M_{(\vec{x}\leftarrow\vec{a})}]\{\vec{a}/\vec{x}\}\varphi \in s$ for all sequences $\vec{a}$ of the right length. Note that $[M_{(\vec{x}\leftarrow\vec{a})}]$ has no existential entry.

Since s is ω-complete, from $\neg[M_{(\vec{x}\leftarrow\vec{a})}]\{\vec{a}/\vec{x}\}\varphi \in s$ it follows that there exists $\vec{b}$ such that $\neg[M_{(\vec{x},\vec{y}\leftarrow\vec{a},\vec{b})}]\{\vec{a},\vec{b}/\vec{x},\vec{y}\}\varphi \in s$. Note that $[M_{(\vec{x},\vec{y}\leftarrow\vec{a},\vec{b})}] \in cOP$.

Let $[A] = [M_{(\vec{x},\vec{y}\leftarrow\vec{a},\vec{b})}]$. One shows that $\Gamma_A = \{\psi \mid [A]\psi \in s\} \cup \{\neg\{\vec{a},\vec{b}/\vec{x},\vec{y}\}\varphi\}$ is consistent and ω-complete following (the proofs of) Theorem 48 and Theorem 52. Also, it follows that there exists a saturated extension s'_A (a state in the canonical model) such that $\Gamma_A \subseteq s'_A$ and, thus, $\neg\{\vec{a},\vec{b}/\vec{x},\vec{y}\}\varphi \in s'_A$. From $\Gamma_A \subseteq s'_A$ and the definition of $R_\mathcal{C}$, we conclude that $(s, s'_A) \in R_\mathcal{C}([\![A]\!]_\mathcal{C})$. ■

We can now prove the *truth lemma*.

THEOREM 54. — *For any state s in $\mathcal{M}_\mathcal{C}$ and any formula φ in the language:*

$$\mathcal{M}_\mathcal{C}, s \models \varphi \textit{ if and only if } \varphi \in s$$

PROOF. — We proceed by induction on the complexity of φ.

1) Let $\varphi = p_0$ be a proposition identifier.
By definition of the canonical model, $\mathcal{M}_\mathcal{C}, s \models p_0$ if and only if $p_0 \in s$.

2) Let $\varphi = \neg\psi$.
$\mathcal{M}_\mathcal{C}, s \models \neg\psi$ if and only if $\mathcal{M}_\mathcal{C}, s \not\models \psi$. By inductive hypothesis, this happens if and only if $\psi \notin s$. Since s maximal and consistent, the latter is equivalent to $\neg\psi \in s$.

3) Let $\varphi = \chi \rightarrow \psi$.
$\mathcal{M}_{\mathcal{C}}, s \models \chi \rightarrow \psi$ if and only if $\mathcal{M}_{\mathcal{C}}, s \not\models \chi$ or $\mathcal{M}_{\mathcal{C}}, s \models \psi$. By inductive hypothesis, this happens if and only if $\chi \notin s$ or $\psi \in s$. Since s maximal and consistent, this is is equivalent to $\chi \rightarrow \psi \in s$.

4) Let $\varphi = [A]\psi$ with $[A] = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \in$ *cOP* $\cup$ *vOP*.

(Left to right) Let $\mathcal{M}_{\mathcal{C}}, s \models [A]\psi$. If $(s, s') \in R_{\mathcal{C}}([\![A]\!]_{\mathcal{C}})$, then $\mathcal{M}_{\mathcal{C}}, s' \models \psi$. By inductive hypothesis, this happens if and only if $\psi \in s'$ for all such s'. By definition of $\mathcal{R}_{\mathcal{C}}$, $\psi \in s'$ for any saturated extension s' of $\{\varphi \mid [A]\varphi \in s\}$. By Theorem 52, there exist $\varphi_1, \ldots, \varphi_r \in \{\varphi \mid [A]\varphi \in s\}$ such that $\vdash (\varphi_1 \wedge \ldots \wedge \varphi_r) \rightarrow \psi$. From Propositions 36 and 37, $\vdash ([A]\varphi_1 \wedge \ldots \wedge [A]\varphi_r) \rightarrow [A]\psi$. We conclude $\{[A]\varphi_1, \ldots, [A]\varphi_r\} \vdash [A]\psi$. From assumption $\varphi_1, \ldots, \varphi_r \in \{\varphi \mid [A]\varphi \in s\}$, it follows $[A]\varphi_1, \ldots, [A]\varphi_r \in s$. Finally, $[A]\psi \in s$.

(Right to left) If $[A]\psi \in s$ and $\{\varphi \mid [A]\varphi \in s\} \subset s'$, then $\psi \in s'$. Since this happens for any s' such that $(s, s') \in R_{\mathcal{C}}([\![A]\!]_{\mathcal{C}})$, we conclude $\mathcal{M}_{\mathcal{C}}, s \models [A]\psi$.

5) Let $\varphi = [M]\psi$ with $[M] = [AB] \in$ *cOP* $\cup$ *vOP*.

[By (inner) induction on the number of columns with basic case 4).]

First, recall from Theorem 48 that for all $(s, s') \in R_{\mathcal{C}}([\![AB]\!]_{\mathcal{C}})$ there exists s'' such that $(s, s'') \in R_{\mathcal{C}}([\![A]\!]_{\mathcal{C}})$ and $(s'', s') \in R_{\mathcal{C}}([\![B]\!]_{\mathcal{C}})$.

Clearly, $\mathcal{M}_{\mathcal{C}}, s \models [AB]\psi$ if and only if for all $(s, s') \in R_{\mathcal{C}}([\![AB]\!]_{\mathcal{C}})$, $\mathcal{M}_{\mathcal{C}}, s' \models \psi$. From above, this holds if and only if for all $(s, s') \in R_{\mathcal{C}}([\![AB]\!]_{\mathcal{C}})$, there exists s'' such that $(s, s'') \in R_{\mathcal{C}}([\![A]\!]_{\mathcal{C}})$, $(s'', s') \in R_{\mathcal{C}}([\![B]\!]_{\mathcal{C}})$, and $\mathcal{M}_{\mathcal{C}}, s'' \models [B]\psi$.

By inner inductive hypothesis, $[B]\psi \in s''$. Furthermore, $\{\chi \mid [A]\chi \in s\} \subset s''$ for all such s'' and so $[A][B]\psi \in s$. From this, $\mathcal{M}_{\mathcal{C}}, s \models [AB]\psi$ if and only if $[A][B]\psi \in s$. By (S^S) and (J^S), $[A][B]\psi \in s$ if and only if $[AB]\psi \in s$, which is to say, $\mathcal{M}_{\mathcal{C}}, s \models [AB]\psi$ if and only if $[AB]\psi \in s$.

6) Let $\varphi = [M]\psi$ with $[M] \in$ *qOP*.

a) Assume $[M]$ without existential entries and let $\vec{x}$ be all its universally quantified variables.

(Left to right) $\mathcal{M}_{\mathcal{C}}, s \models [M]\psi$ if and only if for any sequence $\vec{a}$ of constants (of the right length), if $A = M_{(\vec{x} \leftarrow \vec{a})}$, then $\mathcal{M}_{\mathcal{C}}, s \models [A]\{\vec{a}/\vec{x}\}\psi$. From cases $4)$ and $5)$, for any such A, $\mathcal{M}_{\mathcal{C}}, s \models [A]\{\vec{a}/\vec{x}\}\psi$ if and only if $[A]\{\vec{a}/\vec{x}\}\psi \in s$. Since s is ω-complete, if $[A]\{\vec{a}/\vec{x}\}\psi \in s$ for all $\vec{a}$ of the right length, then $[M]\psi \in s$.

(Right to left) The result follows directly from $(\forall E^S)$, cases $4)$ and $5)$, and the semantic clauses.

b) Assume $[M]$ has existential entries and let $\vec{x}$ be all its existentially quantified variables.

(Left to right) There exist $\vec{a}$ such that $\mathcal{M}_{\mathcal{C}}, s \models [M_{(\vec{x} \leftarrow \vec{a})}]\{\vec{a}/\vec{x}\}\psi$ Now, if $[M_{(\vec{x} \leftarrow \vec{a})}]$ is quantificational we apply case a) above, while cases $4)$-$5)$ are used if

$[M_{(\vec{x}\leftarrow\vec{a})}]$ is uniform. In this way we obtain: $\mathcal{M}_{\mathcal{C}}, s \models [M_{(\vec{x}\leftarrow\vec{a})}]\{\vec{a}/\vec{x}\}\psi$ if and only if $[M_{(\vec{x}\leftarrow\vec{a})}]\{\vec{a}/\vec{x}\}\psi \in s$. From the latter, by applying ($\exists I^{\mathcal{S}}$), one gets $[M]\psi \in s$.

(Right to left) Since s is saturated, there exists $\vec{a}$ such that $[M_{(\vec{x}\leftarrow\vec{a})}]\{\vec{a}/\vec{x}\}\psi \in s$. If $[M_{(\vec{x}\leftarrow\vec{a})}]$ is quantificational, then apply case a) above, otherwise apply cases 4)-5). We obtain $\mathcal{M}_{\mathcal{C}}, s \models [M_{(\vec{x}\leftarrow\vec{a})}]\{\vec{a}/\vec{x}\}\psi$. From the semantics, $\mathcal{M}_{\mathcal{C}}, s \models [M]\psi$.

■

The above results suffice to establish the strong completeness of the logic with respect to multi-agent Kripke frames.

Let K be the class of multi-agent Kripke frames for k. Given $\Sigma \subseteq \mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$, we write $\Sigma \models_{\mathrm{K}} \varphi$ to mean that φ is valid in each model for Σ, then

COROLLARY 55. — *Let $\Sigma \cup \varphi$ be a set of formulas in some language $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ (Σ possibly empty), then*

$$\Sigma \models_{\mathrm{K}} \varphi \textit{ if and only if } \Sigma \vdash \varphi$$

4.3. *Decidability*

Decidability guarantees that there is an algorithm which can answer any question stated in the language. In the area of knowledge representation and reasoning, one of the reasons for the success of the modal approach over the years is that a surprising number of modal logic systems are indeed decidable. We now turn to investigate this issue with respect to our logic $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$. At first, one may be discouraged by the fact that, generally speaking, modal logics extended with first-order quantifiers are not decidable. This is a consequence of the fact that standard first-order logic is already not decidable. However, in our case quantifiers play a particular role in the whole system because of the restriction to occur only within modalities. We will come back to this observation in section 5.

Our goal is to prove that $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ is decidable. More precisely, we show that for any fixed k, there exists an algorithm which (correctly) establishes whether a formula in the language $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ is satisfied in some model for the logic $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$. To reach the conclusion, we will make use of the decidability results for the propositional system *mPDL* (see section 2), the proof of which relies on the finite model property and the filtration technique. In short, given a formula φ of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ we will show how to obtain a variable-free formula ψ whose decidability can be checked in *mPDL*. The result on ψ will allow us to establish the result on φ itself. The overall procedure guarantees the decidability of the logic $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$.

Fix a formula $\varphi \in \mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$. Without loss of generality, we assume that in φ no variable occurs quantified more than once and that a variable cannot have both free and bound occurrences. As a consequence, a quantificational operator occurs at most once in φ. Also, we assume that the number of elements in *ActId* is higher than the number of symbols in φ.

Our first step consists in isolating enough action identifiers to witness free and quantified variables in φ. We write $deg_v(\varphi)$ for the number of free variables in φ, and for each operator $[M]$ occurring in φ we write $deg_\exists(M)$ for the number of existential entries in M. Let $ActId_\varphi = \{a \in ActId \mid a \text{ occurs in } \varphi\}$ and fix a set B such that:

a) $ActId_\varphi \cap B = \emptyset$ and

b) $|B| = deg_v(\varphi) + \Sigma_{M\in\varphi} deg_\exists(M)$
where $M \in \varphi$ stands for "M occurs in φ".

(Note that $ActId_\varphi \cup B$ is finite. Also, it is non-empty whenever φ is modal.)

The idea is to restrict our attention to the language with action identifiers in set $ActId_\varphi \cup B$ only, and to consider a propositional formula that is obtained from φ by using combinations of elements in $ActId_\varphi \cup B$ to instantiate the variables. In the next steps, we show how to construct such a formula.

The function $g_{\varphi,B}$ over the formulas of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ with action identifiers from $ActId_\varphi$, is defined recursively by the following clauses (we omit indices φ, B since no confusion arises):

$$
\begin{array}{rcl}
g(p) & = & p \quad \text{for } p \text{ atomic} \\
g(\neg\psi) & = & \neg g(\psi) \\
g(\psi \to \chi) & = & g(\psi) \to g(\chi) \\
g([M]\psi) & = & [M]g(\psi) \quad \text{for } [M] \text{ uniform} \\
g([M]\psi) & = & \bigwedge_{\vec{b}}[M_{(\vec{y}\leftarrow\vec{b})}]\{\vec{b}/\vec{y}\}g(\psi) \quad \text{for } [M]\ \forall\text{-uniform in } \textit{qOP} \\
g([M]\psi) & = & \bigvee_{\vec{a}}[M_{(\vec{x}\leftarrow\vec{a})}]\{\vec{a}/\vec{x}\}g(\psi) \quad \text{for } [M]\ \exists\text{-uniform in } \textit{qOP} \\
g([M]\psi) & = & \bigvee_{\vec{a}}\bigwedge_{\vec{b}}[M_{(\vec{x},\vec{y}\leftarrow\vec{a},\vec{b})}]\{\vec{a},\vec{b}/\vec{x},\vec{y}\}g(\psi) \quad \text{for the other cases}
\end{array}
$$

where

i) $\vec{x}$ are all existentially quantified variables of M (if any),

ii) $\vec{y}$ are all universally quantified variables of M (if any), and

iii) $\vec{a}, \vec{b} \in ActId_\varphi \cup B$ are of the same length of $\vec{x}$ and $\vec{y}$, respectively.

Finally, let g^* be given by

$$g^*(\psi) = \bigvee_{\vec{c}\,\in\, ActId_\varphi \cup B} \{\vec{c}/\vec{z}\}g(\psi)$$

where $\vec{z}$ collects all the free variables of $g(\psi)$.
If $g(\psi)$ contains no variable, then put $g^*(\psi) = g(\psi)$.

Clearly, $g^*(\varphi)$ does not contain any variable, i.e., it is a formula in the language of *mPDL*. Furthermore, we know that *mPDL* enjoys the finite model property and is decidable (see Proposition 13). Then, we can use filtration to establish whether $g^*(\varphi)$ is satisfiable in a model for *mPDL*, that is, if there exists a state of a *mPDL*-model at which φ is true. Since a model for *mPDL* can be turned into a model for $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ (and vice versa) preserving the truth of the propositional formulas, the same conclusion holds by considering $g^*(\varphi)$ as a formula of $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$. It remains to show that $g^*(\varphi)$ holds at a state of some model if and only if φ does.

PROPOSITION 56. — *$g^*(\varphi)$ is satisfiable if and only if φ is.*

PROOF. — The left to right direction is easy. By assumption, $g^*(\varphi)$ holds at a finite model with $ActId = ActId_\varphi \cup B$. Then, it suffices to read off $g^*(\varphi)$ the action identifiers that serve as instances for the free and existentially quantifed variables in φ. (Regarding the universally quantified variables, note that each disjunct of $g^*(\varphi)$ includes all the instance combinations of elements in $ActId_\varphi \cup B$.) Trivially, φ holds at the very same state in virtue of those action identifiers.

For the right to left direction. Assume φ is satisfiable. We show that $g^*(\varphi)$ is satisfiable as well. (Without loss of generality, below we assume that B and $ActId$ are disjoint sets.)

Fix a state s in the canonical model $\mathcal{M}_\mathcal{C}$ such that φ holds at it.

DEFINITION 57 (INSTANCE OF A FORMULA AT A STATE s). —

A formula ϕ is an instance of a formula ϑ *at state s of a model $\mathcal{M}$, written $\phi \in FInst_{\mathcal{M},s}(\vartheta)$, if the following conditions are satisfied*

1) let ϑ be atomic, then $\phi = \vartheta$,

2) let $\vartheta = \neg\chi$, then $\phi = \neg\gamma$ with $\gamma \in FInst_{\mathcal{M},s}(\chi)$,

3) let $\vartheta = \chi_0 \to \chi_1$, then $\phi = \gamma_0 \to \gamma_1$
with $\gamma_0 \in FInst_{\mathcal{M},s}(\chi_0)$ and $\gamma_1 \in FInst_{\mathcal{M},s}(\chi_1)$,

4) let $\vartheta = [M]\chi$, then $\phi = \{\vec{a}/\vec{x}\}([M_{(\vec{y}\leftarrow\vec{b})}]\{\vec{b}/\vec{y}\}\gamma)$
where
$\vec{x}$ collects all and only the variables free in ϑ (if any);
$\vec{y}$ collects all and only the quantified variables in $[M]$ (if any);
$[\![a_i]\!] = [\![x_i]\!]$ for all i;
$a_i, b_j \in ActId$ for all i, j; and
$\{\vec{a}, \vec{b}/\vec{x}, \vec{y}\}\gamma \in FInst_{\mathcal{M},s}(\chi)$.

From the definition, the number of instances of a formula with variables depends on the action identifiers in the language. Since we concentrate on formula instances in the canonical model, for each action in $Act_\mathcal{C}$ an identifier in *ActId* is available. Also, note that from the definition we have $FInst_{\mathcal{M},s}(\vartheta) \subseteq mPDL$.

Clearly, if φ is satisfiable at a state s of the canonical model, then at least one of its instances is satisfiable at s as well. Let ψ be such an instance. Let C contain all and only actions identifiers that are in ψ but not in $ActId_\varphi$. Note that $|C| \leq |B|$. Thus, we can find a set $B^\circ \subset ActId$ such that $C \subseteq B^\circ$ and $ActId_\varphi \cap B^\circ = \emptyset$, and a bijection

$f : B^\circ \rightarrow B$. Let ψ° be obtained from ψ by substituting each action identifier $a \in B^\circ$ with $f(a)$ and let $\mathcal{M}^\circ$ be obtained from $\mathcal{M}_\mathcal{C}$ by putting

(i) $[\![a]\!]^\circ = [\![a]\!]$ for $a \notin B^\circ$,

(ii) $[\![f(a)]\!]^\circ = [\![a]\!]$ for $a \in B^\circ$.

The remaining changes in $\mathcal{M}^\circ$ are at this point obvious. Informally, we have just renamed a finite number of action identifiers throughout the canonical model. Clearly, $\mathcal{M}^\circ$ is isomorphic to $\mathcal{M}_\mathcal{C}$. Then, ψ° is satisfied in $\mathcal{M}^\circ$ and, by construction, ψ° is a disjunct in $g^*(\varphi)$ and so we are done. ■

We have thus proven the decidability of the logic.

THEOREM 58. — $\mathcal{L}^\mathcal{S}_{\mathcal{MA}(k)}$ *is decidable with respect to* K.

5. $\mathcal{L}^\mathcal{S}_{\mathcal{MA}}$ and normal modal logics

The proof of completeness we presented in section 4.2 is based on a strategy that, when applied to first-order modal logic, relies on the adoption of the *Barcan formulas*. These, in the traditional modal language, correspond to implication

$$(BF) \qquad \forall x \,\Box\, \varphi \rightarrow \Box \forall x \varphi$$

and its converse. Their role is to force the domain of quantification to remain fixed across all states in the frame.

As we already pointed out, quantification has a different purpose in our logic. Nevertheless, our system relies on the assumption that the domain of quantification *Act* does not change if one moves from one state to another in a Kripke frame for multi-agent systems. Since quantification and modality are strictly intertwined in this language and the Barcan formulas are not expressible in the standard way, one may wonder where the axiomatization imposes the corresponding constraint on the set of actions *Act*. Looking at the deductive system, one sees that the role of the Barcan formulas in our multi-agent logic is taken by the *Split* (S^S) and the *Join* (J^S) schemas. Schema (S^S) is strongly related to formula $\forall x \Box \varphi \rightarrow \Box \forall x \varphi$ as one can see comparing the semantics of the two subformulas in $\mathcal{L}^\mathcal{S}_{\mathcal{MA}}$. In short, this schema says that an action available at the initial state, where the $[MN]$ operator is evaluated, is also available at the state where the single operator $[N]$ is evaluated. (J^S) provides the other link. Leaving aside the restrictions (which are due mainly to "information issues" for the agents and not to properties of the states themselves), this latter schema states that it makes no difference to quantify on actions at a later state (when we instantiate $[N]$ of the antecedent subformula) or directly at the initial state (where we instantiate $[MN]$). Semantically, this means that every action available at a later state is already present at the initial one.

More generally, the techniques that allowed us to prove completeness and decidability of $\mathcal{L}^\mathcal{S}_{\mathcal{MA}}$ have been developed for modal normal logics. But, strictly speaking, our logical system is not normal. We restricted the application of axiom schema (K^S) to a subset of the modal operators, namely the set of $\forall$-uniform operators, simply because (K^S) fails when there are existential entries. Nevertheless, this mismatch is only apparent. Looking closely, the applicability of these methods is guaranteed by the fact that the logic $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$, being built on top of the multi-normal propositional logic *mPDL*, has a natural interpretation on frames. The constant operators lead to the adoption of multi-agent Kripke frames and, as a consequence, of the standard techniques available on frames. On this specific issue, the quantificational modal operators have a marginal role. They can be seen as special *sets of constant operators* and, as such, they do not affect the adopted notion of frame on which the operators' interpretation relies.

Frame analysis for $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$

If the above observations are correct, one surmises that other properties of propositional modal logics are preserved in the richer system $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$. We provide a simple example by considering classical frame analysis.

A few formulas of propositional (normal) modal logic have attracted much attention over the years for their relationship with classes of standard Kripke frames as well as for their relevance in applied logics. Among them, the following schemas deserve mention:

$D.\ \Box\varphi \rightarrow \neg\Box\neg\varphi$

$T.\ \Box\varphi \rightarrow \varphi$

$B.\ \varphi \rightarrow \Box\neg\Box\neg\varphi$

$4.\ \Box\varphi \rightarrow \Box\Box\varphi$

$5.\ \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$

In our multi-agent language, these formulas correspond to ($[A] \in cOP$):

$D_m.\ [A]\varphi \rightarrow \neg[A]\neg\varphi$

$T_m.\ [A]\varphi \rightarrow \varphi$

$B_m.\ \varphi \rightarrow [A]\neg[A]\neg\varphi$

$4_m.\ [A]\varphi \rightarrow [A][A]\varphi$

$5_m.\ \neg[A]\varphi \rightarrow [A]\neg[A]\varphi$

It can be shown [CHE 80] that normal modal logic is determined by some special class of standard Kripke frames depending on which of the above formulas is added as an axiom in the logic. Here is the list of axioms and corresponding classes of standard Kripke frames (see [CHE 80] for a complete discussion):

D	–	standard serial frames
T	–	standard reflexive frames
B	–	standard symmetric frames
4	–	standard transitive frames
5	–	standard euclidean frames

These classical results rely on the normality property of the modalities, so they provide an interesting test for our case. Basically, we want to see if these results can be generalized to our quantificational logic. First, we analyze the relationship between serial Kripke frames and the logic $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$ enriched with schema D_m and, from this result, we obtain a general statement for the other cases as well.

Recall that a frame $\langle W, Act; R\rangle$ is *serial* if for each state $s \in W$ and each constant operator $[A]$, the set $\{s' \mid (s, s') \in R([\![A]\!])\}$ is non-empty. Then,

LEMMA 59. — *D_m is valid in (multi-agent) serial Kripke frames.*

PROOF. — Fix a model $\mathcal{M}$ for $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}}$, a state s in it, and an operator $[A] \in cOP$. Assume $\mathcal{M}, s, \Im \models [A]\varphi$. We show that $\mathcal{M}, s, \Im \models \neg[A]\neg\varphi$. By assumption, if $(s, s') \in R([\![A]\!]_{\mathcal{M}})$ then $\mathcal{M}, s', \Im \models \varphi$. Since $\mathcal{M}$ is serial, there exists s' such that $(s, s') \in R([\![A]\!]_{\mathcal{M}})$. Then, for such s' we have $\mathcal{M}, s', \Im \not\models \neg\varphi$, i.e., $\mathcal{M}, s, \Im \not\models [A]\neg\varphi$. We conclude that $\mathcal{M}, s, \Im \models \neg[A]\neg\varphi$. ■

LEMMA 60. — *Let $\mathcal{M}^{\mathcal{D}}_{\mathcal{C}}$ be the canonical model for $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}} \cup D_m$ (for a given k). Then, $\mathcal{M}^{\mathcal{D}}_{\mathcal{C}}$ is serial.*

PROOF. — We need to show that for every saturated set s (where consistency is now stated with respect to $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}} \cup D_m$) and constant operator $[A]$, there exists a saturated set s' such that $\{\varphi \mid [A]\varphi \in s\} \subseteq s'$. For this, it suffices to establish that $\Gamma = \{\varphi \mid [A]\varphi \in s\}$ is consistent with respect to $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}} \cup D_m$. If not, $\Gamma \vdash_{\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}} \cup D_m} \perp$. Then, there exist formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_{\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}} \cup D_m} (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \perp$. Using (K^S), (Nec), and propositional logic (see Theorem 4.48), we have $\vdash_{\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}} \cup D_m}$ $([A]\varphi_1 \wedge \ldots \wedge [A]\varphi_n) \to [A]\perp$. Then, by the application of (MP) and D_m, one shows $\vdash_{\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}} \cup D_m} ([A]\varphi_1 \wedge \ldots \wedge [A]\varphi_n) \to \neg[A]\neg\perp$. Since Γ contains $\varphi_1, \ldots, \varphi_n$, then $\{[A]\varphi_1, \ldots, [A]\varphi_n\} \subset s$, i.e., $\neg[A]\neg\perp \in s$. By (Nec), $[A]\neg\perp \in s$ also. Thus, s is not consistent, contradicting the assumption that s is saturated. ■

We have just proven that schema D_m is valid in serial Kripke frames and that the canonical model has a serial Kripke frame. The proof of this result is obtained by a simple adaptation of the completeness proof of the previous section analogously to what happens for standard modal logic. Also, it is clear that quantificational operators play no role in the new steps, namely Lemmas 59 and 60, introduced to force the canonical model to be constructed over a serial frame. The rest of the completeness proof is also unaffected by D_m and the new constraint on frames.

The above lemmas on D_m show how to modifying the proofs given in [CHE 80] to deal with the remaining axiom schemas. In this way, one verifies that the above result is not an exception, it holds in all the listed cases. Furthermore, the result yields decidability for the enriched logics since filtration preserves the properties we are dealing with (see [CHE 80]).

Now, we can formally state our result.

THEOREM 61. — *The logic $\mathcal{L}^{\mathcal{S}}_{\mathcal{MA}(k)}$ augmented with schema $D_m(T_m, B_m, 4_m, 5_m)$ is complete and decidable with respect to the class of serial (respectively: reflexive, symmetric, transitive, euclidean) multi-agent Kripke frames for k.*

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