Atomicity vs. Infinite Divisibility of Space

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Abstract. In qualitative spatial reasoning, the last ten years have brought a lot of results on theories of spatial properties and relations taking regions of space as primitive entities. In particular, the axiomatization of mereotopologies has been extensively studied. However, properties of space such as divisibility, density and atomicity haven't attracted much attention in this context. Nevertheless, atomicity is especially important if one seeks to build a bridge between spatial reasoning and spatial databases approaches in areas like vision or GIS. In this paper we will investigate the possibility of characterizing such properties in spaces modeled by mereologies and mereotopologies. In addition, properties of atoms like extension and self-connectedness will be considered.

Keywords: ontology of space, qualitative spatial reasoning, axiomatic theories of space, spatial data representation.

1 Introduction

From ancient philosophy on, the question of the nature of space and its constituting elements has played a great part in metaphysical studies. This question raised some problems that still make up interesting research topics not only in philosophy but in mathematics and cognitive sciences as well. What does constitute space? The Greeks took for granted that perceptible entities occupying space (for example, physical objects) are extended bodies, i.e., they have a *magnitude*. Having a magnitude meant to the Greeks these entities can be divided: «every extended magnitude is divisible» [12]. But can division be repeated infinitely or not? Are ultimate constituents, i.e., indivisible entities, reached by this division process? Would such "indivisibles" be without magnitude, i.e., points, or should we consider extended one-piece basic constituents, i.e., atoms? In the first case, the problem faced by the Greeks was to accept that extended entities be constituted by entities without magnitude, even an infinite number of them. In the second case, the problem was to give up the divisibility principle of any extended magnitude. In a second stage, additional questions regarding the relationships holding between basic constituents, if any, are raised. Is there always a third entity between two of them (density)? If so, how can a magnitude be "continuous", i.e., one-piece, if these basic entities cannot touch? Or else, is there always a next one, touching it, with no other in between (discreteness)? Are these constituents in a finite or infinite number?

In spite of the interest of these issues, our intentions are not to get into philosophical analyses, but rather to investigate all the possible options in the context of spatial representation and spatial reasoning in AI. In particular, we will examine the atomic option, that is, a discrete space built on one-piece extended basic entities. This one seems particularly adequate for the representation of space in several contexts. For instance, since our perceptual and cognitive capacities are finite and finiteness implies discreteness, if cognitive space contains some notion of density, it is probably a density “in intension”. As a matter of fact, linguistic descriptions of everyday space do not imply the existence of an infinite domain. But above all, finiteness is a reality in computer science, and no matter what we’d like to represent, physical space or cognitive space, it is doomed to be in a finite way. Moreover, several areas of computer science (e.g., vision, GIS, qualitative modeling) are dedicated to handling discrete spaces with only finite numbers of spatial entities (pixels, regions, positions).

In this paper we will investigate the possibility of characterizing and imposing the properties discussed above in axiomatic theories of space used in qualitative spatial reasoning (QSR hereafter), more precisely in mereologies and mereotopologies which have played a great role in the development of QSR in the last ten years [13, 8, 1, 3, 11]. Atomic theories could constitute a direct means of exploiting reasoning in spatial databases. Surprisingly enough, very little attention has been paid to atomicity and cardinality properties in mereotopologies, even though some problems related to these issues had already been identified in [13].

2 A Solution of an Ancient Puzzle: the Concept of Atom

In their search for the ultimate parts that constitute space[1] and the matter that occupies it, the Greeks generally agreed on the following facts:

a) Extension: what we can perceive are magnitudes, i.e., extended bodies
b) Divisibility: a magnitude is divisible into two magnitudes
c) Self-connectedness: a magnitude is one-piece, or “continuous”.

The famous paradoxes of Zeno show that they couldn’t accommodate all these facts easily. For them, carrying fact b) to the limit seemed incompatible with a) and c): any number, even infinite, of 0-dimension entities would not constitute an extended magnitude, and since two points cannot touch without being equal, any sum of points would be holed. In addition, after Pythagoras, the Greeks knew that there are missing “points” between the points built by dividing up a magnitude, in modern words, irrationals separating the rationals. Attempts to solve this problem have been numerous [16, 12]. We report only four important ones below.

Aristotle's Solution. Aristotle held that one can always divide a magnitude any finite number of times but that infinite divisibility is only potential. This interpretation of b) solves the problem, but does so by renouncing to the search about the nature of the constituting elements of space: there are just no such things as ultimate parts.

Euclid's Solution. In the first book of The Elements (on plane geometry), Euclid circumvents the problem by introducing points, lines and surfaces, as distinct primitive terms (“real” definitions) which are related by incidence relations, not constitution relations. Thus the problem of magnitudes having no ending point for being incommensurable with those built by division is avoided, eluding the question

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1 It is not clear whether all ancient philosophers conceived space as a containing void, i.e., for several of them, magnitudes could have been only material objects. The extrapolation to the nature of an absolute space is ours.

of the ultimate constituents of space. Infinity is also only potential, in agreement with Aristotle: a line (segment) might be prolonged at will but doesn't exist in its entirety.

The Continuum Solution. In the second half of the 18^{th} century, Cantor and Dedekind found out that there are distinct infinities, i.e., cardinals of infinite sets that cannot be put in a one-to-one correspondence. They also discovered that infinite sets of 0-dimensional entities could indeed make up a “continuum”, i.e., a continuous extended magnitude. *The* continuum, the real line, is built through Dedekind's completion of the rationals with the irrationals, as it were filling the holes between them. Therefore, b) can be carried to the limit (and in fact further) without contradicting a) and c) and the concept of transformation founding the modern approach of geometry may be introduced.

The results of analytic geometry and transformation theory are extensively applied in computer science to solve a great number of problems that can be described quantitatively, i.e., in numeric terms. To deal with qualitative information, QSR has developed an alternative approach taking extended entities (also called regions) as primitives instead of the customary sets of 0-dimensional points. Among the three solutions above, Aristotle's is the only one in the spirit of the region approach. Another one, the atomic solution, clearly adopts non-divisibility: atoms are ultimate constituents, which, as regions, are extended.

The Atomic Solution. The Atomists claimed that the divisibility of magnitudes no longer holds under a certain grain. All matter[2] is made up of *atoms* i.e., indivisible magnitudes, touching or surrounded by the void. This answer solved Zeno's paradox by limiting the application of b) and enabled the atomists to account for the possibility of motion. But what exactly “indivisible magnitude” means? Intuitively, the notion of divisibility is the ability to break up an entity into two components. Thus, defining atoms as indivisible entities implies the impossibility to find two smaller parts that make it up. By the preceding definition, if we are able to find one smaller part only, this entity would be indivisible. One could think we are overlooking the fact that if there is a smaller part, then there is necessarily another one, making the difference between the part and the whole. As we will see in Section 3, this is true when decomposition is considered to be extensional, but not in the general case. As a result, a more generic notion of atom has been considered in philosophy and mathematics. An atom simply does not have any smaller part, i.e., it has no proper parts (or proper subsets). Under this view, a 0-dimensional point is also an atom. This more generic notion not only changes indivisibility for the no-proper part property, but also drops the notion of extension, the notion of atoms as magnitudes.

As we will see in next sections, in mereologies it is possible to define divisibility, the simple no-proper part condition, and the fact that atoms constitute space, i.e., everything is a sum of atoms. Topological concepts in mereotopologies make possible the introduction of notions of extension and “one-pieceness”. Unfortunately, finiteness of the space is not first-order definable, and tiling theory shows that in a space of 2 or more dimensions, discreteness may be defined only using a metrics [9], which is out of the scope of the present paper. Still, a special attention will be given to the possibility for the theories considered to model tessellations (discrete partitions of a connected space in which the “tiles” are extended), since spatial data as images or country maps are all tessellations of a portion of the plane.

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[2] In the Middle Ages, this solution has been extended to an atomic account of container space.

3 Atomicity, Divisibility and Density in Mereology

Basic Mereology and its Extensions. Mereology is a theory of the binary relation P (for part), originally introduced by Lesniewski in [10] as an alternative to set theory. Recently, it has been used both in formal ontology, to model the generic part-whole relation [14, 20, 15], and in QSR, to model spatial inclusion between regions of space taken as primitive entities [13, 1, 3, 11]. In this paper, the latter point of view is of interest to us.

From basic mereology, noted **M**, a variety of stronger theories can be contemplated. This section, as well as Section 4.1 reviewing mereotopology, is based on two excellent works, [14] and [20]. The notation is principally inspired by that of [20]. We chose P as primitive relation. In addition to those of first-order logic with equality, **M** consists of the following three axioms:

$$P(x,x) \tag{P1}$$
$$(P(x,y) \wedge P(y,x)) \rightarrow x=y \tag{P2}$$
$$(P(x,y) \wedge P(y,z)) \rightarrow P(x,z) \tag{P3}$$

In this theory, the relations of proper part (PP) and overlap (O) are defined by:

$$PP(x,y) \equiv_{def} P(x,y) \wedge \neg P(y,x) \tag{DPP}$$
$$O(x,y) \equiv_{def} \exists z\, (P(z,x) \wedge P(z,y)) \tag{DO}$$

From **M**, two classes of extensions can be made. First, one may require the extensionality of PP adding to **M** the following axiom (EXT) and call the resulting theory **WM**, for weak extensional mereology (**WM**=**M**+(EXT)).

$$(\exists z\, PP(z,x) \wedge \forall z\, (PP(z,x) \rightarrow PP(z,y))) \rightarrow P(x,y) \tag{EXT}$$

In **WM** the following formula is a theorem, proving that PP is extensional:

$$(\exists z\, PP(z,x) \wedge \forall z\, (PP(z,x) \leftrightarrow PP(z,y))) \rightarrow x=y \tag{T1}$$

However, there might be two different entities a and b, a being the only proper part of b, without any other part of b making the difference between a and b. Adding the supplementation axiom (SUP) to **M** yields **EM**, the theory called extensional mereology in the literature (**EM**=**M**+(SUP)).

$$\neg P(x,y) \rightarrow \exists z\, (P(z,x) \wedge \neg O(z,y)) \tag{SUP}$$

(SUP) is stronger than (EXT), i.e., **EM** $\vdash$ **WM** and **WM** $\nvdash$ **EM** [14].

It is interesting to note that in **EM**, but not in **WM**, O is extensional:

$$\mathbf{EM} \vdash \forall z\, (O(z,x) \leftrightarrow O(z,y)) \rightarrow x=y \tag{T2}$$

The second way of extending **M** is to impose the existence of sums, products and differences (whenever they are not empty). In a first stage, one may require only the existence of binary sums and products, yielding what Varzi calls closed mereology (**CM**=**M**+(SUM)+(PROD)+(DIF)).

$$\exists z\, \forall w\, (O(w,z) \leftrightarrow (O(w,x) \vee O(w,y))) \tag{SUM}$$
$$O(x,y) \rightarrow \exists z\, \forall w\, (P(w,z) \leftrightarrow (P(w,x)\ \&\ P(w,y))) \tag{PRO}$$
$$\exists z\, (P(z,x) \wedge \neg O(z,y)) \rightarrow \exists z\, \forall w\, (P(w,z) \leftrightarrow (P(w,x)\ \&\ \neg O(w,y))) \tag{DIF}$$

Sometimes an extra axiom ensuring the existence of the universe is added. This entity is unique by (P2); we will note it U:

$$\exists x\ \forall y\ P(y,x) \tag{UNI}$$

If the domain is infinite, one may want to add the existence of sums and products of an infinite number of entities with an axiom stating that the fusion of any non-empty set of entities[3] exists, yielding general mereology (**GM**=**M**+(FUS)).

$$\exists x\ \phi(x) \rightarrow \exists z\ \forall y\ (O(y,z) \leftrightarrow \exists x\ (\phi(x) \wedge O(y,x))) \tag{FUS}$$

The two directions of extension may be combined. The strongest of the theories obtained is **GEM**, general extensional mereology, also called classical mereology in the literature. We obtain the lattice on Figure 1.

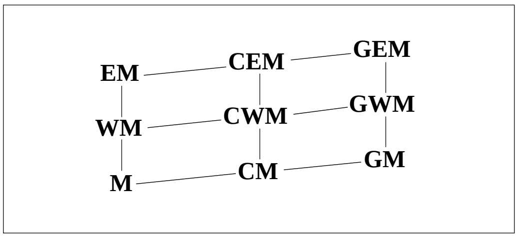

Fig. 1. The lattice of the mereologies

(T2) being a theorem only in **EM**, the unicity of the entities whose existence is asserted by (SUM) and (FUS) is only guaranteed where (SUP) holds. In other words, the operators of sum (+) and fusion (σ) may be introduced in **CEM** and **GEM** respectively, but not in weaker mereologies. For their part, the operators of product (•) and difference (-) may already be introduced in **CM**, unicity being guaranteed by (P2)[4]. These operators are usually introduced with Russell's description operator ι[5]:

$$x+y =_{def} \iota z\ \forall w\ (O(w,z) \leftrightarrow (O(w,x) \vee O(w,y))) \tag{D+}$$
$$x \bullet y =_{def} \iota z\ \forall w\ (P(w,z) \leftrightarrow (P(w,x)\ \&\ P(w,y))) \tag{D•}$$
$$x\text{-}y =_{def} \iota z\ \forall w\ (P(w,z) \leftrightarrow (P(w,x)\ \&\ \neg O(w,y))) \tag{D-}$$
$$\sigma x \phi(x) =_{def} \iota z\ \forall y\ (O(y,z) \leftrightarrow \exists x\ (\phi(x) \wedge O(y,x))) \tag{Dσ}$$

We will use the notation ~x for the complement of x, that is, U-x, which exists in **CM**+(UNI)[6] and in **GEM** provided there is some y not overlapping x:

$$\sim x =_{def} \iota z\ \forall y\ (P(y,z) \leftrightarrow \neg O(y,x)) \tag{D~}$$

The models of **GEM** have been characterized by Tarski [18]. **GEM** is proved to be complete with respect to the class of the complete quasi-Boolean algebras, i.e., complete complemented distributive lattices with the null element removed.

Atoms and Atomicity. As discussed in Section 2 the generic notion of atom (i.e. not considering splitting-divisibility and the extension of atoms) may be characterized in basic mereology **M** with two equivalent definitions:

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3 Characterized in the following axiom schemata by the formula $\phi(x)$ in which x is free but not y nor z. Note that $\phi(x)$ may be any first-order formula, not only a formula written in the language of mereology.
4 No other versions of (SUM) and (FUS) in terms of P could yield unicity without (SUP).
5 Individuals introduced exist only under the conditions required by (PRO), (DIF) and (FUS).
6 But notice that ~~x=x holds only in **CEM**+(UNI).

$$\mathrm{ATOM}(x) \equiv_{def} \forall y\ \neg PP(y,x) \qquad \text{(DA1)}$$
$$\mathrm{ATOM}(x) \equiv_{def} \forall y\ (P(y,x) \rightarrow y=x) \qquad \text{(DA2)}$$

The existence of atoms may be introduced by:

$$\exists x\ \mathrm{ATOM}(x) \qquad \text{(AT0)}$$
$$\forall x\ \exists y\ (\mathrm{ATOM}(y) \wedge P(y,x)) \qquad \text{(AT1)}$$
$$\forall z\ (\mathrm{ATOM}(z) \rightarrow (P(z,x) \rightarrow P(z,y))) \rightarrow P(x,y) \qquad \text{(AT2)}$$

Axiom (AT0) ensures the simple existence of at least one atom, whereas the atomicity axiom (AT1) guarantees the everywhere existence of atoms. (AT2) is “atomic essentialism”.

[20] shows that (AT0) and (AT1) are independent[7] from **GEM**, and consequently from any weaker mereology. (AT2) is also independent from **GEM**.

Supplementation is implied by (AT2) (but not by (AT0) nor (AT1)):

$$\text{(AT2)} \vdash \text{(SUP), thus } \mathbf{M}\text{+(AT2)} \vdash \mathbf{EM} \qquad \text{(T3)}$$

Figure 2.a shows the relationships between (AT0), (AT1) and (AT2). The semantics of this graph is as follows. For each arrow, the axiom at the starting node added to the set of axioms labeling the arrow entails the formula at the ending node. In addition, if we replace a mereology labeling an arrow by a weaker one according to the lattice of Figure 1, we do not get the entailment, i.e., the label is “minimal”.

Since (AT2) implies **EM**, and in **EM** (AT1) and (AT2) are equivalent, in the remainder of this paper, we will consider only the addition of (AT0) and especially (AT1) to mereologies.

Divisibility. Axiom (DV0) ensures the existence of at least one non-atomic region, i.e., a region in which there are no atoms. The “real” atomlessness axiom is (DV1).

$$\exists x\ \forall y\ (P(y,x) \rightarrow \exists z\ PP(z,y)) \qquad \text{(DV0)}$$
$$\forall x\ \exists y\ PP(y,x) \qquad \text{(DV1)}$$

As discussed in Section 2, to get infinite divisibility, we need more than (DV1). (DV1) is satisfied by a model in which there is a single infinite chain of nested proper parts. With (DV2), there are at least two non-overlapping proper parts in each entity:

$$\forall x\ \exists yz\ (PP(y,x) \wedge PP(z,x) \wedge \neg O(y,z)) \qquad \text{(DV2)}$$

Axiom (DV3) corresponds more closely to what the Greeks were considering, namely that each entity is divisible into two halves:

$$\forall x\ \exists y,z\ (x=y+z \wedge \neg O(y,z)) \quad \text{(in } \mathbf{CEM} \text{ or } \mathbf{GEM} \text{ only)} \qquad \text{(DV3)}$$

[20] shows that (DV0) and (DV1) are independent from **GEM** and any weaker mereology. This is the case also for (DV2) and (DV3).

Figure 2.b shows the relationship between (DV0), (DV1), (DV2) and (DV3). (DV1) and (DV2) are equivalent from **EM** up, but not in other mereologies. Moreover from **CEM** up, (DV1), (DV2) and (DV3) are equivalent. Since **CEM** is the minimal theory in which (DV3) makes sense, we will not consider it any longer[8].

Regarding the relationships between atomicity and divisibility we can observe that axioms (AT0) and (DV0) are compatible in any mereotopology so that a “mixed”

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[7] An axiom A is independent from a theory **T** iff $T \nvdash A$ and $T \nvdash \neg A$.

[8] Similarly, $\mathrm{ATOM}(x) \equiv_{def} \neg\exists y,z\ (x=y+z \wedge \neg O(y,z))$ would be an atom definition closer to the Greeks' one than (DA1-2). In **CEM**, it is equivalent to (DA1-2).

space with both atoms and non-atomic entities is possible. However, their negations are incompatible as soon as there is one entity: (AT0) and (DV1) are incompatible, as are (DV0) and (AT1) and any pair (ATn) and (DVm) for n,m≥1.

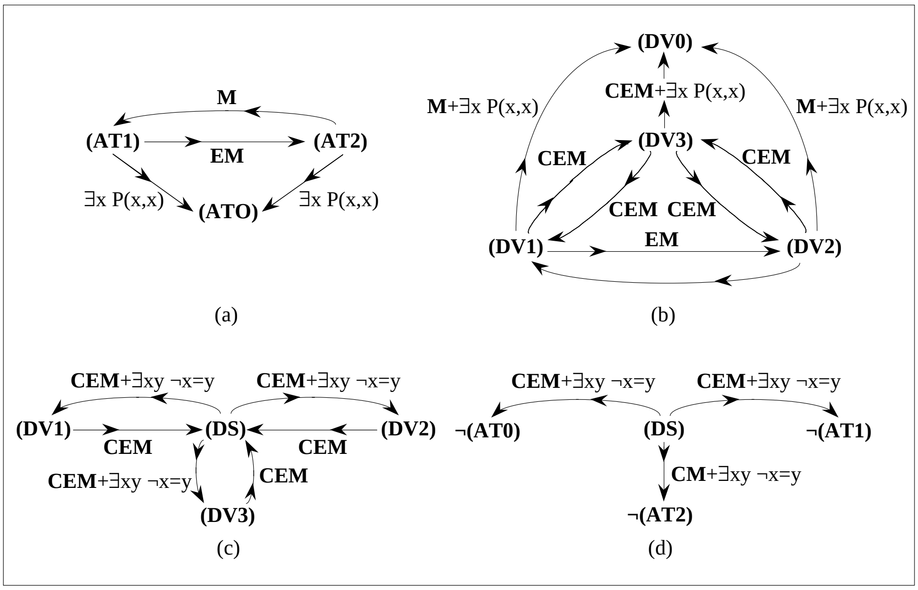

Fig. 2. Relationships between different existence axioms for atomicity, divisibility and density in the mereologies

Density. Mathematically speaking, density is a property of orders[9], generally taken to be linear orders. Since P is a partial order, we can introduce a notion of "density" in mereologies in an analogous way. The following has been proposed in [21, 14, 1]:

$$\forall xy\ (PP(x,y) \rightarrow \exists z\ (PP(x,z) \wedge PP(z,y))) \qquad \text{(DS)}$$

This axiom is independent from all mereologies.

Figure 2.c shows the relationships between density and divisibility. From **CEM** up, (DS) is equivalent to any of the divisibility axioms (DVi).

Figure 2.d makes the relationships between density and atomicity explicit.

4 Atomicity, Divisibility and Density in Mereotopologies

4.1 Mereotopology

Theories combining mereological notions and topological ones like those of "being connected with", "being an interior part of" or "being self-connected" have been called mereotopologies. There are at least two strategies to axiomatize such theories [20]. In the first, one extends a mereology with an added topological primitive [19, 4, 3]. In the second strategy, one considers particular theories in which both

[9] Given a set S equipped with a total order relation ≤, S is *dense* iff $\forall x,y\in S\ (x<y \rightarrow \exists z\ x<z<y)$

mereological and topological notions can be axiomatized with a unique primitive [5, 13, 1]. Since the theories developed in the second strategy correspond to particular cases of those of the first, we will adopt the first and try to discuss most of the theories which have been proposed in the literature.

Basic Mereotopology. The chosen new primitive added to P is the binary relation C (for connection) which reads "is connected with"[10]. As in [20], we will call basic mereotopology (**MT**) the theory obtained by adding to **M** the following three axioms (**MT**=**M**+(C1)+(C2)+(C3)):

$$C(x,x) \tag{C1}$$

$$C(x,y) \rightarrow C(y,x) \tag{C2}$$

$$P(x,y) \rightarrow \forall z\, (C(z,x) \rightarrow C(z,y)) \tag{C3}$$

Three useful predicates, EC (external connection), IP (interior part), and IPP (interior proper part) are defined:

$$EC(x,y) \equiv_{def} C(x,y) \wedge \neg O(x,y) \tag{DEC}$$

$$IP(x,y) \equiv_{def} P(x,y) \wedge \neg\exists z\, (EC(x,z) \wedge EC(y,z)) \tag{DIP}$$

$$IPP(x,y) \equiv_{def} PP(x,y) \wedge IP(x,y)) \tag{DIPP}$$

With axioms (IP) or (IPP), we will be able to characterize the spaces in which all elements are "extended", i.e., have an interior part[11]:

$$\forall x\, \exists y\, IP(y,x) \tag{IP}$$

$$\forall x\, \exists y\, (IPP(y,x)) \tag{IPP}$$

Basic Strong Mereotopology. Many authors require axiom (C4), the converse of (C3), to strengthen the links between P and C and so further constrain the "spatiality" of their interpretation. **SMT**=**MT**+(C4) is basic strong mereotopology.

$$\forall z\, (C(z,x) \rightarrow C(z,y)) \rightarrow P(x,y) \tag{C4}$$

In this theory, connection becomes extensional:

$$\mathbf{SMT} \vdash \forall z\, (C(z,x) \leftrightarrow C(z,y)) \rightarrow x=y \tag{T4}$$

With (C3) and (C4) in strong mereotopologies, P is an unnecessary primitive. Indeed, several authors propose theories based on a unique primitive C [1,5,13]. **SMT** is equivalent to a theory based on C alone with (C1-2) and (T4) as axioms, and with a definition of P (in terms of C) instead of (C3)+(C4).

Surprisingly enough, imposing extension with (IP) in **EM** yields (C4). When (IP) holds, **EMT** = **SEMT**:

$$\mathbf{EMT}\text{+(IP)} \vdash \text{(C4)} \tag{T5}$$

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[10] Different possible interpretations for C are investigated in [7].

[11] Formally, characterization is a property of a theory with respect to a class of models. We (sloppily) use this word with respect to an intended interpretation of our primitives in classical topological spaces. Under this intended interpretation, introducing an axiom discards some undesired models. But without a completeness proof, this intended interpretation is not guaranteed. Here, the notion of "extension" is meant to correspond to the topological one: $Ext(A) \equiv_{def} i(A) \neq \emptyset$, where i is the interior operator of the topological space.

Non Basic Mereotopologies. We can of course consider stronger mereologies instead of **M**, and obtain corresponding mereotopologies named **XT** or **SXT** where **X** stands for any mereology of Section 3.

The property of self-connectedness[12], Con, can be characterized in mereotopologies in which the sum is definable, i.e., in **CEMT** or higher:

$$Con(x) \equiv_{def} \forall yz\ (x=y+z \rightarrow C(y,z)) \qquad \text{(DCon)}$$

However, this definition is not really operative for atoms, because it is trivially verified and thus does not exclude any undesired model. We will consider in Section 5 whether self-connectedness can be characterized "externally", through the relationships of each atom with its surrounding ones.[13]

In **SMT**, the extensionality of C (T4) enables the introduction of operators alternative to +, -, σ and ~, in whose definitions C replaces O. We will use primes to distinguish the new operators and note **SCMT'** or **SGMT'** the mereotopologies build with corresponding existence axioms (SUM') and (DIF') or (FUS')[14]. Similarly, we will note Con' the property of self-connectedness using +' instead of +.

$$\exists z\ \forall w\ (C(w,z) \leftrightarrow (C(w,x) \vee C(w,y))) \qquad \text{(SUM')}$$
$$\exists z\ (P(z,x) \wedge \neg C(z,y)) \rightarrow \exists z\ \forall w\ (P(w,z) \leftrightarrow (P(w,x)\ \&\ \neg C(w,y))) \qquad \text{(DIF')}$$
$$\exists x\ \phi(x) \rightarrow \exists z\ \forall y\ (C(y,z) \leftrightarrow \exists x\ (\phi(x) \wedge C(y,x))) \qquad \text{(FUS')}$$

Comparing (T4) with (T2) reveals how different **SCEMT'** and **SGEMT'** are from **SCEMT** and **SGEMT**. In the prime versions of the definitions (and contrarily to the original ones), what occurs "at the boundaries" of the entities introduced, i.e., how they are externally connected to others, contributes to determine their identity.

With these prime operators, it can be noted that, whenever they exist, the difference between y and x and the complement of x are not connected to x: $\forall xy\ (\neg C(y\text{-}'x,x) \wedge \neg C(x,\sim' x))$. Therefore, Con(U) brings an inconsistency in **SCMT'**. This fact has led some authors, notably Cohn and his co-authors in [13] to introduce another complement operator which directly implies the connection between any region and its complement, and thus the self-connectedness of the universe Con(U).

$$\sim_{rcc}x =_{def} \iota y\ \forall z\ ((C(z,y) \leftrightarrow \neg IPP(z,x)) \wedge (O(z,y) \leftrightarrow \neg P(z,x))) \qquad \text{(D}\sim_{rcc}\text{)}$$
$$x\text{-}_{rcc}y =_{def} \iota w\ \forall z\ ((C(z,w) \leftrightarrow C(z, x\bullet(\sim_{rcc}y))) \qquad \text{(D-}_{rcc}\text{)}$$

In addition, this definition of complement yields another interesting result. **SCMT'rcc** (**SCMT'** with the complement and the difference operators defined as in (D$\sim_{rcc}$) and (D-$_{rcc}$)) entails (SUP) so that **SCMT'rcc**= **SCEMT'rcc**.

In the rest of this paper, we will consider mereotopologies where operators may be properly defined, that is, mereotopologies at least as strong as **CEMT**, **SCMT'** or **SCEMT'rcc** and in which (UNI) holds.

The Open-closed Distinction. Topology is a theory based on the notion of open set. Here, we have introduced mereotopologies without referring at all to this arguably non-commonsensical notion, a fact which is considered an asset by several authors [13, 3]. What would be an open region in a mereotopology? Just as in classical

[12] In a topological space <S,T> with the closure and interior operators (c and i), a subset A of S is *self-connected* if it is not the union of two separated parts, that is, formally, iff: $\forall B,C \subset A\ (A=B\cup C \rightarrow (c(B)\cap C \neq \emptyset \vee B\cap c(C) \neq \emptyset))$.

[13] Even though (IP) seems to refer to some internal structure, it is still operative for atoms, because IP(x,x) imposes the non-connection of an atom with any other.

[14] (UNI) and its alternative, (UNI'), are equivalent in **SMT**, the minimal context where (UNI') could be considered with some interest.

topology, a region equal to its interior. We may have interior parts, but the interior doesn't necessarily exist. When one wants to make the difference between open and closed regions, one may impose with the following axiom (INT) the existence of the interior ix of any region x having at least one interior part:[15]

$$\exists w\ IP(w,x) \rightarrow \exists y\ \forall z\ (O(z,y) \leftrightarrow \exists w\ (IP(w,x) \wedge O(w,z))) \qquad \text{(INT)}$$
$$ix =_{def} \iota y\ \forall z\ (O(z,y) \leftrightarrow \exists w\ (IP(w,x) \wedge O(w,z))) \qquad \text{(Di)}$$

The unicity of the interior is guaranteed by the extensionality of O in **CEMT**. Notice that (INT) is redundant where (FUS) holds. Following the definitions of classical topology, closure (c) and boundary (b) operators[16] are introduced. The properties of being open (Op) and closed (Cl) are also defined in a classical manner:

$$cx =_{def} \sim i\sim x \quad \text{(or } c'x =_{def} \sim' i' \sim' x) \qquad \text{(Dc)}$$
$$bx =_{def} cx\text{-}ix \quad \text{(or } b'x =_{def} c'x\text{-}'i'x) \qquad \text{(Db)}$$
$$Opx \equiv_{def} x=ix \quad \text{(or } Op'x \equiv_{def} x=i'x) \qquad \text{(DOp)}$$
$$Clx \equiv_{def} x=cx \quad \text{(or } Cl'x \equiv_{def} x=c'x) \qquad \text{(DCl)}$$

In theories in which the operators are based on C, the presence of the closure operator enables the definition of another notion of self-connectedness, Con2, with which Con2(U) doesn't entail inconsistency in **SCMT'**+(UNI):

$$Con2(x) \equiv_{def} \forall yz\ (x=y+'z \rightarrow C(cy,cz)) \qquad \text{(DCon2)}$$

Where Con2 is expressible, it is weaker than Con: $\forall x\ (Con(x) \rightarrow Con2(x))$ because we always have $\forall x\ P(x,cx)$.

To get a complete topological picture, something is still missing. We need to impose that the product of two open regions is open as well:

$$(O(x,y) \wedge Op(x) \wedge Op(y)) \rightarrow Op(x\bullet y) \quad \text{(or similarly with Op')} \qquad \text{(OPP)}$$

We'll add the letter **O** to denote mereotopologies with the open/closed distinction and the open product condition above: **COEMT**=**CEMT**+(INT)+(OPP), **GOEMT**=**GEMT**+(OPP), **SCOMT'**=**SCMT'**+(INT')+(OPP'), etc. This open/closed distinction is not always operative. First, if there is no interior part in any element, axiom (INT) doesn't impose the existence of anything. Together with axioms (IP) and (UNI) the existence of all interiors and closures is guaranteed. However, the distinction collapses in some contexts:

$$\textbf{S(CO/G/GO)EMT}\text{+(IP)+(UNI)} \vdash x=ix=cx \qquad \text{(T6)}$$
$$\textbf{S(CO/G/GO)EMT'}\text{+(IP)+(UNI)} \vdash x=i'x=c'x \qquad \text{(T7)}$$

Then, **S(CO/GO)EMT** becomes equivalent to **S(C/G)EMT**, and similarly for the prime versions. Lastly, introducing (INT) may render some theories inconsistent as soon as there is an interior part in some element, i.e., as soon as there are two non-connected elements. This happens with the family of **Xrcc** mereotopologies. Notice that this shows also that (FUS) cannot be added to these theories. As a result, the only remaining theory in this family worth to be considered is **SCEMT'rcc**, which is precisely the one presented in [13] (with (IPP) as an additional axiom).

Figure 3 recapitulates the different mereotopologies to be considered.

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[15] With the extensionality of C in **SCMT'**, C replaces O in these formulas yielding (INT'), (Di'), etc.

[16] These definitions don't show the conditions of existence of closures and boundaries. cx or c'x exists whenever x has a complement (i.e., when x is not the universe) and this complement has an interior part. bx (or b'x) exists whenever ix and cx exist and there is a part of cx that doesn't overlap (in O-based theories), or connect (in C-based ones) ix.

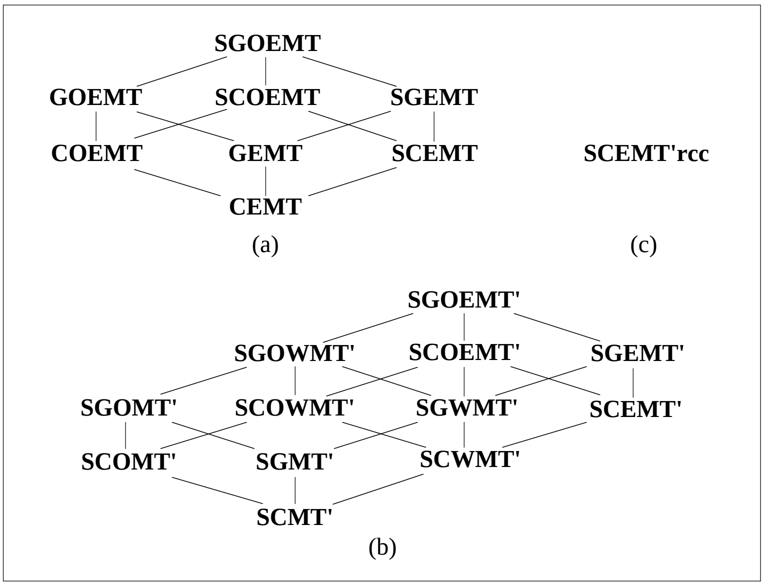

Fig. 3. The different lattices of mereotopologies

4.2 Consistency of Mereotopologies with Atomicity and other Constraints

In addition to the existence of the universe, of (at least) finite sums, differences and products, two further assumptions are made in the rest of this paper[17]. First, that there are at least two distinct elements in the domain so the theories are not trivial. Second, that the universe is self-connected[18]. This means either Con(U) or Con'(U) depending on the choice of O-based or C-based operators. Where interior and closure operators are introduced, we will also consider Con2(U) or Con2'(U).

Under these assumptions, we examine which mereotopologies are consistent or inconsistent with the addition of atomicity (AT1), divisibility (DV1-2) or density (DS) axioms. When atomicity holds, it is interesting to see whether the extension axiom (IP) may be added whereas in the case of divisibility we will consider the stronger (IPP). According to Figure 3, we consider three classes of mereotopologies:

a) mereotopologies with O-extensionality and operators defined in terms of P/O.
b) mereotopologies with C-extensionality and operators defined in terms of C.
c) RCC theories.

Before giving the results in the tables below, here's a simple fact:

Fact 1: Given **X**+(A) $\vdash$ (B), if **X**+(A) is consistent, then **X**+(B) too, and if **X**+(B) is inconsistent then **X**+(A) too.

According to the lattices in Figure 3 and Fact 1, we will only discuss the maximal consistent theories and the minimal inconsistent ones under additional hypotheses.

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[17] In our opinion, these hypotheses are not over-limiting. They find a natural interpretation in spatial data like images and maps.

[18] The same results are obtained considering only the maximal connected parts of the universe, provided these parts are not all atomic, i.e., that the topology is not discrete.

Table 1. Consistent (▲) and inconsistent (▽) mereotopologies under atomicity. Exploiting the lattices of Figure 3 and Fact 1, only the maximal consistent and minimal inconsistent theories are indicated in each case.

	(AT1)		(AT1)+(IP)	
	Con(U)	Con2(U)	Con(U)	Con2(U)
a	▲**GOEMT** ▽**SCEMT**	▲**GOEMT** ▽**SGEMT** ▽**SCOEMT**	▽**CEMT**	▽**COEMT** ▽**GEMT**
b	▽**SCMT'**	▲**SGOEMT'**	▽**SCMT'**	▲**SGOWMT'** ▽**SGEMT'** ▽**SCOEMT'**
c	▽**SCEMT'rcc**	—	▽**SCEMT'rcc**	—

Table 2. Consistent (▲) and inconsistent (▽) mereotopologies under divisibility and density.

	(IPP)		(DV1-2)∨(DS)	
	Con(U)	Con2(U)	Con(U)	Con2(U)
a	▲**SGOEMT**	▲**SGOEMT**	▲**SGOEMT**	▲**SGOEMT**
b	▽**SCMT'**	▲**SGOWMT'** ▽**SGEMT'** ▽**SCOEMT'**	▽**SCMT'**	▲**SGOEMT'**
c	▲**SCEMT'rcc**	—	▲**SCEMT'rcc**	—

Atomicity (Table 1)

a) GOEMT+(AT1)+Con(U) is consistent[19], and similarly with Con2(U) because of Con(U)→Con2(U) and Fact1. Notice that in these cases, the "topological" operators i and c are not classical because there are models in which they behave as "erosion" and "expansion" operators of pretopology[20] [2] and all the elements in the domain are neither open nor closed. When (C4) holds, (AT1) and Con(U) entail that atoms, having to be connected to their complements, are part of them. Thus **SCEMT**+(AT1)+Con(U) is inconsistent. One could think that, in models in which atoms are disconnected to their complements, we could have Con2(U). However, in this case, x=ix=cx holds for all atoms, therefore the self-connectedness of the universe is impossible, and **S(CO/G)EMT**+(AT1)+Con2(U) are inconsistent. **CEMT**+(AT1)+(IP)+Con(U) and **(CO/G)EMT**+(AT1)+(IP)+Con2(U) are also inconsistent because, by (T5), (C4) holds, so we fall back into the previous cases.

b) SGOMT'+(IP), Clarke's theory [5], is consistent with (AT1) and Con2'(U)[21] and there, topological operators behave classically. The stronger **SGOWMT'**+(AT1)+

[19] The following structure is a model of **GOEMT**+(AT1)+Con(U): <S,P,C> where S={∪X | X⊆I} with I={[n,n+1] ⊂ ℝ, n∈ℤ}, C(x,y) iff x∩y≠Ø and P(x,y) iff x⊆y. An analogous structure with a finite domain is a model too (compatibility with finitude).

[20] For example, IP(ix,ix) and ccx=cx are not theorems.

[21] A class of models is given in [1]. Note that finite models exist.

(IP)+Con2'(U) is also consistent, although its models are rather limited[22]. Without (IP), we can have (SUP): **SGOEMT'**+(AT1)+Con2'(U) is consistent[23], but here again, topological operators are not classical. However, with (IP), **S(CO/G)EMT'**+(AT1)+(IP)+Con2'(U) are inconsistent because i) if, for some x, its boundary bx exists, bx must have an interior part which will then overlap ix, which is impossible by definition (Db), and ii) if there are no boundaries, we have ∀x x=ix=cx, so that Con2'(U) boils down to Con'(U) and, as mentioned in Section 4.1, with (C4), ∀x ¬C(x, ~'x). This last remark shows also why **SCMT'**+Con'(U) is inconsistent.

c) **SCEMT'rcc**+(AT1) is inconsistent, as shown in [13].

Divisibility and Density (Table 2)

a) In this case all the theories are consistent. **SGEMT**+(IPP)+Con(U) is consistent[24], but the open/closed distinction collapses (because of (T6) and (IPP)→(IP), by definition) and thus **SGOEMT**+(IPP)+Con(U) is consistent as well. Because of Fact 1 and Con(U)→Con2(U), the same holds for **SGOEMT**+(IPP)+Con2(U). Since (IPP) entails (DV1) by definition and, moreover, in **CEM** (DV1), (DV2) and (DS) are equivalent, **SGOEMT**+(DV1-2/DS)+ Con(U)/Con2(U) are consistent.

b) Clarke's theory is consistent with (IPP) and Con2'(U) [6] so that **SGOMT'**+(IPP)+Con2'(U) is consistent. **SGOWMT'**+ (IPP)+Con2'(U) is also consistent but models are similar to those for the atomicity case[25]. Without (IPP), it is possible to have (SUP) and models with boundaries and classical topological operators: **SGOEMT'**+(DV1-2/DS)+Con2'(U) is consistent[26]. As for corresponding cases with (AT1)+(IP), and for the same reasons, **S(CO/G)EMT'**+(IPP)+Con2'(U) are inconsistent. Lastly, we note again that **SCMT'**+Con'(U) is already inconsistent.

c) **SCEMT'rcc**+(IPP) is consistent and entails Con'(U)[27]. As in case a) above, we get the consistency of **SCEMT'rcc**+(DV1-2/DS)+Con'(U) too.

5 Discussion: Towards an Atomic Theory of Space

Let's try now to draw some conclusions with respect to the general objective of this paper: modeling tessellations in a mereotopological framework (see Section 2).

We are left with two classes of mereotopologies consistent with (AT1):

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[22] It is impossible to have more than one external connection for each entity. The following structure is a model for **SGOWMT'**+(AT1)+(IP)+Con2'(U): <S,P,C> where S={[-1,0], [0,1],]-1,0[,]0,1[⊂ ℝ}, P(x,y) is x⊆y and C(x,y) is x∩y≠Ø.

[23] We can obtain a model for **SGOEMT'**+(AT1)+Con2'(U) changing the domain of the structure of footnote 19 for S'={∪X | X⊆I and if {[n,n+1],[n+2,n+3]}⊆X then [n+1,n+2]∈X and if {[n,n+1],[n+3,n+4]}⊆X then [n+1,n+2]∈X and [n+2,n+3]∈X}. In this case too, finite models exist.

[24] A model is given by the non-empty regular open sets in the standard topology of $\mathbb{R}^2$ interpreting P as set inclusion and C as the non-empty intersection of the closures.

[25] The following structure is a model for **SGOWMT'**+(IPP)+Con2'(U): <S,P,C> where S= $\bigcup_{n=1}^{+\infty}\bigcup_{i=1}^{2^{n-1}}\left\{\left]\frac{2i-2}{2^n},\frac{2i-1}{2^n}\right],\left[\frac{2i-1}{2^n},\frac{2i}{2^n}\right[\right\}$, P(x,y) is x⊆y and C(x,y) is x∩y≠Ø.

[26] A model is given in the standard topology of $\mathbb{R}^2$, by a set of open concentric rings, all arcs of concentric circles with no ending points, and all sums of these, interpreting P as set inclusion and C(x,y) as cx∩y≠Ø ∨ x∩cy≠Ø.

[27] A class of models is given in [17].

(O-)Extensional Mereologies with Non-classical Topological Operators. The fact that pretopologies are models of these theories may be seen as an asset, depending on the particular objectives sought (e.g. operators of erosion and expansion are useful for image processing). In these theories the entities of the domain are all of the same topological kind, neither closed nor open, which is actually the case for the tiles in tessellations. Theories without (C4) present the advantage of having atoms directly connected with their complement, whereas in those with (C4), atoms are isolated. More importantly perhaps, in all these theories, it is not possible to introduce the "extension" axiom (IP).

Non-(O-)extensional Mereologies with Classical Topological Operators. In this case, (C4) holds and it is possible to add (IP) to characterize the extension of atoms. However, atoms are open and isolated. To obtain the self-connectedness of the universe, each atom necessarily comes along with an additional different entity, its closure. In other words, each atom x is "duplicated" into a pair <x=ix, cx>, an ontologically controversial fact. Moreover, we cannot have (SUP) here and the difference between each atom and its closure doesn't belong to the domain. Even though **SGOWMT'** is consistent with the weaker (EXT), its models are limited (cf footnote 22) so we have to use **SGOMT'** at most.

What can be done at this point to obtain a theory that gets closer to our intended models? If we believe O-extensionality to be the most important, we have two open problems at hand. i) Without (C4), one could believe P is not constrained enough (with respect to the topological primitive C) to account for spatial inclusion. Thus, the possibility to add weaker axioms than (C4) should be explored. ii) Without (IP), a different way to characterize the extension of atoms should be looked for.

If, on the other hand, we insist on capturing some features of classical topology in a domain of extended atoms and choose the second option, we could try to introduce a kind of second-level theory that would "filter" the domain of the mereotopology retaining only open entities (alternatively, only closed ones). An axiom of O-extensionality limited to these entities could then be introduced in the mereotopology without leading to an inconsistency:

$$(Op(x) \land Op(y) \land \neg P(x,y)) \rightarrow \exists z\, (Op(z) \land P(z,x) \land \neg O(z,y)) \qquad \text{(SUPop)}$$

We will not follow this way further here, so let's explore what can be done to solve i) and ii) with the first option.

Figure 4.a, which is a model of **GEMT** but not **SGEMT** with its intuitive interpretation, shows why we cannot have (C4): everything connected to a is also connected to b and vice-versa, but we want a and b to be different. Actually, in the counter-model 4.a, we have not even used (C4) but a weaker property: theorem (T4). When trying to weaken (C4) into any of the following three axioms, the resulting theory is still inadequate as shown on Figure 4 by models 4.a (with (C4''), a=a+b) and 4.b (with (C4'), a=b, and with (C4'''), a+b=b).

$$\forall z\, (EC(z,x) \leftrightarrow EC(z,y)) \rightarrow x=y \qquad \text{(C4')}$$

$$(\forall z\, (C(z,x) \leftrightarrow C(z,y)) \land P(x,y)) \rightarrow x=y \qquad \text{(C4'')}$$

$$(\forall z\, (EC(z,x) \leftrightarrow EC(z,y)) \land P(x,y)) \rightarrow x=y \qquad \text{(C4''')}$$

Could we add some other axiom to **GEMT** to link P and C further and still remain compatible with atomicity and extensionality of O? This question is still open at the moment. In case the answer should be negative, we may wonder whether we need to add another spatial primitive to get closer to our intended models. This new question seems to have a positive answer if we consider another desired properties for atoms, self-connectedness, as well as extension, our second problem at hand. Figure 4.c and

4.d show that it is impossible, in general, to characterize the self-connectedness of atoms in the mereotopological language under our intended interpretation (in which cell-adjacency in supposed to be encoded by EC). It is not possible to discard model 4.c because in both figures (4.c and 4.d), the grayed region (considered as an atom) has exactly the same relationships with respect to all the surrounding atoms.

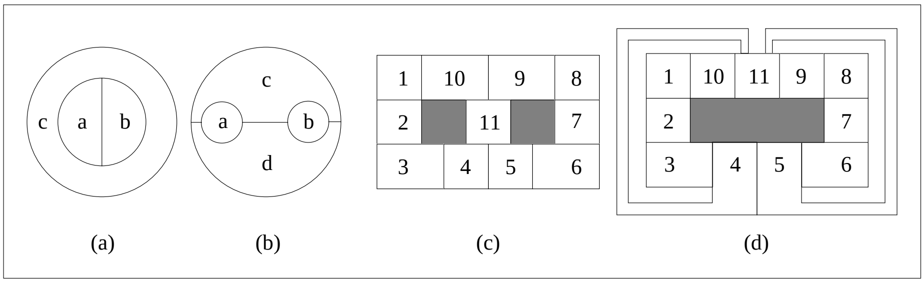

Fig. 4. Inadequacy of the extensionality of C for atomic theories and impossibility to account for the self-connectedness of atoms

Similarly, we can see on Figure 5.a and 5.b that both extended atoms and non-extended ones may be in the exactly same configuration. The same examples show that introducing the distinction between point-connection and line-connection doesn't help. We are thus inclined to believe that a new primitive, probably of a morphological flavor, is required to fully formalize these notions.

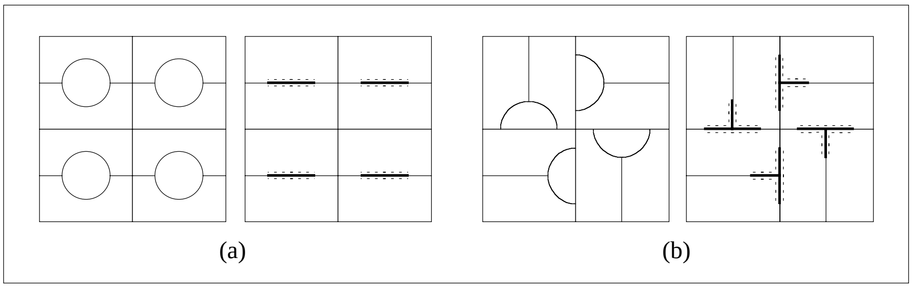

Fig. 5. Impossibility to account for the extension of atoms

So, what about the shape of atoms? Here again, describing the shape of an atom requires referring to its relationships with surrounding atoms. Unfortunately, tiling theory tells us that such a description would be very limited. In some regular plane tilings, only the minimum and maximum number of neighbors of each tile can be determined, even applying to them heavy restrictions (like the congruence between tiles) [9]. Anyway, dealing with morphological notions on atoms is not at all obvious. Indeed, it is on their incapacity to build a theory agreeing with Euclidean geometry that the atomists have been the most seriously criticized in the past [12].

To conclude, these results show that the atomic structures we were alluding to in the introduction are not characterizable by a mereotopology. Nevertheless we hope that the systematic analysis accomplished in this paper will be useful as a basis for future work along this direction.

Acknowledgements

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