

# FULL MEREOMETRIES

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**ABSTRACT.** We analyze and compare geometrical theories based on mereology (*mereometries*). Most theories in this area lack in formalization and this prevents any systematic logical analysis. To overcome this problem, we concentrate on specific interpretations for the primitives and use these to isolate comparable models for each theory. Relying on the chosen interpretations, we introduce the notion of environment structure, that is, a minimal structure containing a (sub)structure for each theory. In particular, in the case of mereometries, the domain of an environment structure is composed of particular subsets of  $\mathbb{R}^n$ . The comparison of mereometrical theories within these environment structures shows dependencies among primitives and provides (relative) definitional equivalences. With one exception, we show that all the theories considered are equivalent in these environment structures.

## 1 INTRODUCTION

At the time Lobachevskij published [34], the axiomatic foundation of geometry was based on points. Such a formal system, called *Euclidean geometry*, falls short of satisfying cognitive concerns since it aims at modeling physical space relying on the abstract notion of point. The matter in dispute is that human experience of space is experience in magnitude and points cannot be empirically experienced. This simple observation makes evident the need for cognitively and philosophically sound geometrical systems whose formal study began in the nineteenth century (although it has received less emphasis with respect to the contemporary and orthogonal research on the fifth Euclidean axiom).

Taking *solids* as basic entities in his system, Lobachevskij revolutionizes the foundations of geometry from the ontological viewpoint and shows how to fill the gap between geometrical and spatial entities. As it happens often with revolutionary approaches, the work of Lobachevskij is quite obscure and it is

presented only informally. One has to wait almost a century to find a formal presentation of the new approach.

The theories developed by Whitehead [60] (see also Biacino and Gerla [6, 7] for a formal characterization of the theory of Whitehead), De Laguna [20], Nicod [39], Tarski [56], and Grzegorzczak [30]) aim at showing that the concept of point is not necessary in the foundation of geometry and, consequently, that the conceptualization of space can differ on several aspects: properties of the space (e.g. Euclidean vs. non Euclidean geometries), primitive relations (e.g. being aligned, equidistance), and ontological nature of entities.<sup>1</sup> From the viewpoint of geometrical construction, these approaches revolutionize the classical method of defining regions as sets of points. Indeed, here points are treated as particular sets of regions.<sup>2</sup> Since the new theories succeed in defining Euclidean entities and relations within a different logical domain, one cannot rely on purely formal arguments to establish which entities and relations deserve the role of geometrical primitives. Euclidean geometry is now challenged at the level of the basic entities and external considerations start leaking into the geometry paradise.

The new geometries are justified by questions that arise outside the geometrical formalism itself and provide formal theories *adequate* for different tasks. In particular, region-based geometries seem to be cognitively more sound than point-based geometries since they make possible a direct mapping from empirical entities and laws to theoretical entities and formulas. Several issues need to be considered: the consequences of choosing extended regions as primitive entities, the meaning of experiencing empirically extended regions, the doubts about perfect regions<sup>3</sup>. Following De Laguna, one wonders what it means to consider points to be sets of solids. Does it follow that the concept of point is defined in empirical terms? Idealised regions seem closer to empirical experience than Euclidean points, still they already require a form of abstraction. Once we admit this, it is not clear where to stop. Then, why should we reject the usual notion of point? De Laguna is aware of this problem: “Although we perceive solids, we perceive no abstractive set of solids (...) In accepting the abstractive set, we are as veritably going beyond experience as in accepting the solid of zero-length” ([20], pg. 460).

Beginning with the work of Clarke [10, 11], theories based on extended entities have attracted much interest for both their formal aspects<sup>4</sup> and their applicative potentialities. The ontological clearness and the evident connection with physical entities justify the philosophical interest in these theories. This approach receives particular emphasis in the field of *formal ontology*. Here, one assumes the relations of *parthood* and *connection* to be basic notions that are exemplified by spatial or material entities like physical objects, chunks of matter, holes, etc. (see Simons [51], Casati and Varzi [9], and Smith [52]). Nowadays, one refers to these theories as *mereotopologies* since they are characterized by the combination of mereology (based on parthood) and topology (based on connection). Following this

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<sup>1</sup> Instead of talking of *solids*, these authors refer to *regions*, *bodies*, or *volumes*. In some cases these notions are not deeply characterized, thus it is difficult to understand if they presuppose a different intuition about physical objects and their possible locations in space. In this paper, *entity* and *region* are taken as generic and intuitive notions. In addition, some authors have recently developed theories based on domains containing entities of different dimensions, e.g. points, lines, surfaces. See Gotts [29] and Galton [25, 27].

<sup>2</sup> In particular, points are often defined as filters of regions. For a detailed discussion on the construction of points in mereotopology see Biacino and Gerla [6].

<sup>3</sup> Tarski defines points as classes of concentric spheres. However, in nature we don't find perfectly spherical objects. One can argue similarly about fractal-shaped regions or regions with infinitely oscillating boundary (see [43] for a discussion on 'pathological' regular regions).

<sup>4</sup> See Gerla [28] for a good survey of mathematical research in this area. An analysis in terms of lattices is given in Stell [54], and in terms of algebras in Düntsch [24].

terminology, we call *mereometries* the theories that aims to reconstruct geometry extending mereotopological systems.

Mereometries are used in various areas. In Schmidt [49] *physics* is presented as a theory based on extended entities. This theory allows to refer explicitly to the objects involved in experiments. Generally speaking, *cognitive science* and *computational linguistics* analyze the possibility of formalizing human learning, conceptualization, and categorization of spatial entities and relations. In particular, Renz and al. [32, 47] take into account the cognitive adequacy of topological relations while Aurnague et al. [2] and Muller [36] show how mereometrical notions are central in the semantics of natural language. Donnelly [21] formalizes the theory of De Laguna in the perspective of *common-sense analysis* of spatial concepts. In *computer science* and more specifically in *qualitative spatial representation and reasoning* (see Cohn and Hazarika [16] and Vieu [59] for good overviews), mereometries are applied for modeling qualitative morphology and movement of physical bodies (Bennett et al. [3, 4, 5], Borgo et al. [8], Cristani et al. [18], Dugat et al. [23], Muller [37], Galton [26], Li et al. [33], Randell et al. [45, 46]), for describing geographical spaces and entities in *Geographical Information Systems* (Pratt-Harmann and Lemon [41], Pratt-Hartmann and Schoop [43], Stock [55]), as well as for characterizing *medical* and *biological information* (Schulz [50], Cohn [14], Smith and Varzi [53], Donnelly [22]).

In all these areas, specific foundational and applicative concerns affect the development of the theories based on geometrically extended entities. Indeed, in the literature there are numerous mereometries which differ on primitive entities, formal properties, as well as general principles. One surmises that this variety of systems has motivated a plethora of results on their relative strengths and drawbacks. Surprisingly, reading the literature, one cannot find an extended study of the relationships among these systems. Although extensive discussions on the cognitive, linguistic, and philosophical motivations for a theory are often undertaken, these are not accompanied by more formal considerations. The few arguments brought forward to discuss the relative expressive power are limited to previous versions of very similar theories and cannot be generalized to a broader class of systems. Such a lack of comparative analysis has practical reasons, in particular the poor axiomatization of most mereometries.

This being the situation, in the following sections we try to fill the gap presenting a method to systematically compare classes of mereometries. In our intentions, this method allows a comparisons of formal theories while concentrating on the meaning of the primitives. We consider it as a first answer to the need of assessment in this area. One can use this approach to state equivalences and similarities among the theories, in this way facilitating both reuse and communication among different applications. Since this task cannot be undertaken with the standard logical machinery (as mentioned above, most of the mereometries available in literature are only weakly formalized) the method we propose overcomes this problem by giving prominence to conceptual and ontological issues, issues which are at the center of the systems we are interested in.

The goal is to make formally explicit important differences like description completeness and conceptual incompatibilities. Beside the direct advantage provided by a reliable classification method, such a comparison would help in selecting theories according to the applicative or theoretical tasks one is facing, in developing new theories, and in extending or modifying those already available.

## 2 CONCEPTUAL COMPARISON

Generally speaking, logical theories are compared at the syntactic or at the semantic level. In the first case, one focuses on the interdefinability of primitive relations assumed in the theories to prove the

equivalence of their axiomatics; while in the second case, one compares domains and relations of (classes of) models for these theories. Systematic analyses of these kinds have been developed on mereotopologies: Casati and Varzi [9], Simons [51], Masolo and Vieu [35] consider the syntactic level, while Biacino and Gerla [6, 7], Asher and Vieu [1], Pratt-Hartmann and Schoop [42, 44], Roeper [48], and Stell [54] focus on the semantic level.

This kinds of comparison make sense if the theories are well characterized, i.e. the given axiomatization captures the *intended models* and so the *intended meanings* of the primitives. In the case of mereogeometries only two theories have been proved to be semantically complete with respect to models expressed in terms of  $\mathbb{R}^n$ . Donnelly [21] provided a full axiomatization for the theory of De Laguna [20], which is based on the primitive *can connect*, whereas the theory of Tarski [56], based on the primitives of *parthood* and *being a sphere*<sup>5</sup>, is fully axiomatized by Bennett in [3].<sup>6</sup> The other mereogeometries available in literature are only ‘partially’ axiomatized (they are not completely characterized with respect to the intended models), see [2, 8, 13], or are axiomatized only indirectly relying on point-based axioms [39].<sup>7</sup>

In order to overcome this lack of explicit or direct formalization and to carry out an exhaustive and informative analysis of the links between the different theories, we follow the approach delineated for the mereotopologies by Cohn and Varzi in [17] and compare the mereogeometries on the basis of their *intended models*. Cohn and Varzi take classical topology as a unifying framework for the comparison. In the case of mereogeometries we rely on  $\mathbb{R}^n$  since this system is generally used by the authors to describe the (intended) models of their theories.<sup>8</sup>

Some authors describe the intended models in a formal way while, in other cases, the models are only sketched. Therefore, our first task is to isolate interpretations in  $\mathbb{R}^n$  that conform with the formal and informal descriptions and that are compatible with the given axiomatizations. We call any such interpretation a *natural interpretation* and the underlying models are dubbed *natural models* of the theory. These notions are discussed below.

Our second (and main) task is to compare these natural/intended models within the chosen unifying framework, i.e.,  $\mathbb{R}^n$ . The analysis of the models (see section 3.2) reveals that they differ significantly on the primitive predicates adopted while the domains of interpretation, henceforth called *natural domains*,

<sup>5</sup> We use *sphere* for the mereogeometrical notion and *ball* for its intended meaning. That is, the interpretation of a sphere in  $\mathbb{R}^n$  is a set  $ball(c, r) = \{x \in \mathbb{R}^n \mid dist(x, c) < r\}$  for some  $c \in \mathbb{R}^n$  and  $r \in \mathbb{R}^+$ .

<sup>6</sup> Tarski himself axiomatizes the primitives only indirectly. He first defines several relations among spheres (e.g. concentricity), relying on the intended interpretation of the primitives, and provides axioms only for parthood. Then, he introduces points as classes of concentric spheres. In this way, he can define equidistance among points using properties of concentric spheres and adopt the Euclidean axioms to constrain equidistance and, indirectly, the predicate *being a sphere*.

<sup>7</sup> Nicod considers the primitives *parthood* and *conjugation* (from which he defines points and their standard relationships) and assumes all theorems of the point-based Euclidean geometry as axioms to force the desired interpretation for the two primitives. He does not provide a direct set of axioms for the chosen primitives. Nicod is aware of the formal drawbacks of this approach. His main goal is to show that extended regions can be taken as the fundamental entities of geometry and the method he applied does the job. As a result, the system has no proper axiomatization.

<sup>8</sup> This is not in contrast with the ontological nature of mereogeometries because  $\mathbb{R}^n$  is used only as an ‘environment’ for intended models. Indeed, these rely on regions in  $\mathbb{R}^n$  and not on single points. In addition, spatial theories adequate to cognitive or applicative tasks focus on qualitative relations and do not aim at capturing ‘new’ notions of space.

are quite similar. Indeed, all these domains are contained in the class of non-empty *regular regions*.<sup>9</sup> As a consequence, most of our work concentrates on the relationship among primitives. Technically speaking we will proceed as follows: first, we collect all the primitives, say  $P_1, \dots, P_n$ , in the systems we want to compare. Then, for each primitive  $P_i$  we fix an interpretation  $R_i$  in the class  $D$  of regular regions of  $\mathbb{R}^m$  (for some fixed  $m$ ). Keeping the  $R_i$ 's fixed, we define several *environment structures*  $\langle D_j, R_1^j, \dots, R_n^j \rangle$  where  $R_i^j$  is the restriction of  $R_i$  to  $D_j \subseteq D$ . We write  $\Phi$ , without indices, for the most inclusive environment structure, namely  $\Phi = \langle D, R_1, \dots, R_n \rangle$ .

Given an environment structure  $\Phi_j = \langle D_j, R_1^j, \dots, R_n^j \rangle$  and a mereogeometry  $\mathbf{T}$ , we define the *local structure*  $\Phi_j(\mathbf{T})$  to be the structure obtained from  $\Phi_j$  by dropping the sets  $R_i^j$  which are not the semantic counterpart of primitives in  $\mathbf{T}$ , i.e.  $\Phi_j(\mathbf{T})$  is the ‘projection’ of  $\Phi_j$  on the primitives of  $\mathbf{T}$ . For example, if  $P_1$  and  $P_3$  are the only primitives of  $\mathbf{T}$ , then  $\Phi_j(\mathbf{T}) = \langle D_j, R_1^j, R_3^j \rangle$ . Similarly, we write  $\Phi_j(\mathbf{T}+\mathbf{T}')$  for the structure obtained from  $\Phi_j$  by dropping the sets  $R_i^j$  which are neither the semantic counterpart of primitives in  $\mathbf{T}$  nor in  $\mathbf{T}'$ , i.e. the projection of  $\Phi_j$  on the union of primitives of  $\mathbf{T}$  and  $\mathbf{T}'$ . These local structures furnish the backbone of our comparison. As a consequence, our strategy is structure dependent and constitutes a generalization of more traditional comparison methods.

Using environment structures, one can formalize the notion of *conceptually equivalent theories*. This is the motivation for the definitions below. Let  $\mathbf{T}, \mathbf{T}'$  be two mereogeometries:

**Definition 2.1** — *If  $P$  is a primitive of  $\mathbf{T}$ , we say that  $P$  is explicitly  $\Phi_j$ -definable in  $\mathbf{T}'$  if there exists an expression  $\varphi$  in the language of  $\mathbf{T}'$  such that the interpretations of  $P$  and of  $\varphi$  are equivalent in the local structure  $\Phi_j(\mathbf{T}+\mathbf{T}')$ . Expression  $\varphi$  is called a  $\Phi_j$ -definition of  $P$  in  $\mathbf{T}'$ .*

**Definition 2.2** — *A theory  $\mathbf{T}$  is a  $\Phi_j$ -subtheory of  $\mathbf{T}'$  if every primitive  $P$  of  $\mathbf{T}$  has an explicit  $\Phi_j$ -definition in  $\mathbf{T}'$ .*

**Definition 2.3** — *Two theories  $\mathbf{T}$  and  $\mathbf{T}'$  are  $\Phi_j$ -equivalent if  $\mathbf{T}$  is a  $\Phi_j$ -subtheory of  $\mathbf{T}'$  and  $\mathbf{T}'$  is a  $\Phi_j$ -subtheory of  $\mathbf{T}$ .*

**Definition 2.4** — *Let  $\mathbf{T}$  and  $\mathbf{T}'$  be theories with natural domains  $D_i$  and  $D_j$ , respectively. We say that  $\mathbf{T}$  and  $\mathbf{T}'$  are conceptually equivalent if they are both  $\Phi_i$ -equivalent and  $\Phi_j$ -equivalent.*

The notions of  $\Phi_j$ -equivalent and conceptual equivalence call attention to the domains of interpretation and to the expressive power of the systems. In our terminology, two theories are  $\Phi_j$ -equivalent if, roughly speaking, when interpreted in the domain  $D_j$  their primitives have the same expressive power. Now, assume we have a first-order translation between two  $\Phi_j$ -equivalent theories. Given a deductively complete axiomatization of the first theory in  $D_j$ , this furnishes a complete axiomatization of the second theory as well (of course, such an axiomatization is relative to the given domain  $D_j$ ).

In the case  $\mathbf{T}$  has natural domain  $D_i$  and  $\mathbf{T}'$  has natural domain  $D_j$ , the fact that they are  $\Phi_i$ -equivalent and  $\Phi_j$ -equivalent tells us that  $\mathbf{T}$  is a true *conceptual counterpart* of  $\mathbf{T}'$  (and vice versa) since one theory captures the natural model(s) of the other when it is interpreted over the corresponding domain. Finally, note that the notion of conceptual equivalence is independent from the overall set of theories one is considering, that is, from the overall environment structure. Indeed, the inclusion (or exclusion) of other theories does not alter the results about  $\mathbf{T}$  and  $\mathbf{T}'$ .

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<sup>9</sup> A subset  $A$  of  $\mathbb{R}^n$  is said to be a regular region if (a) the closure of  $A$  equals the set obtained by the topological closure of the biggest open set in  $A$  and (b) the interior of  $A$  equals the biggest open set contained in the topological closure of  $A$  itself, see section 3.1.

Some mereogeometries already furnish definitions that aim to capture primitives of other theories. In these cases, it is crucial to verify whether the defined relations really correspond to those primitives. For example, the theory in [21] (theory **T4** of section 3), defines the relation *connection* (**C**) in terms of the primitive *can-connect* (**CCon**) as follows:

$$(2.1) \quad \mathbf{C}(x, y) =_{\text{def}} \forall z(\mathbf{CCon}(z, x, y)).$$

Since **C** is a primitive in the theory of [13] (theory **T6** of section 3), its interpretation must agree with the interpretation obtained by (2.1). In this case, one must verify that the interpretation of **C** given by (2.1), which depends on the interpretation of **CCon**, conforms with the natural interpretation of **C** given by [13] in the domains associated to these theories. A crucial step in our comparison is to provide this kind of analysis.

We hasten to point out that this method is not universal and not always straightforward. Sometimes it is hard to isolate a meaningful environment for comparison or it might turn out that a complete comparative analysis is too complex to be carried out. Some issues based on these considerations are discussed in section 4. Also, it is important to take into account that existential axioms (taken to constrain the domain of interpretation for a given theory) might fail in environment structures with restricted domains.

In the next section we give a description of the mereogeometries studied in this paper together with their natural models expressed in  $\mathbb{R}^n$ . In section 4, we fix and justify our choice of environment structures and, in section 5, we present the details of the comparison verifying the explicit syntactic translations across the theories and introducing new or corrected translations whenever necessary.

### 3 MEROGEOMETRIES

In this section, we present the mereological systems considered in this paper and fix their formal interpretations. Since  $\mathbb{R}^n$  is the common underlying framework, we begin by listing some standard relations of this system and then use them to interpret the mereogeometrical vocabulary.

#### 3.1 BASIC NOTIONS IN $\mathbb{R}^n$

Here we recall some topological and geometrical relations and functions on  $\mathbb{R}^n$ , these are needed for interpreting mereogeometry:

- topological operators: *closure* ( $\square$ ), *interior* ( $^\circ$ );
- Euclidean distance, *dist*:  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, +\infty)$ ;
- standard operators, functions, and relations definable from these.

Also, recall that a subset  $A$  of  $\mathbb{R}^n$  is said to be a *regular region* whenever  $[A]^\circ = A^\circ$  and  $[A^\circ] = [A]$ .

In the following list of operators and relations, lower-case variables stand for points of  $\mathbb{R}^n$ , upper-case variables for regular regions in  $\mathbb{R}^n$ :

Operators and functions in  $\mathbb{R}^n$  ( $X$  and  $Y$  non-empty):

$$\begin{aligned} \text{ball}(c, r) &= \{x \mid \text{dist}(x, c) < r\}, \text{ where } r > 0; & (\text{non-empty open ball of radius } r, \text{ center } c) \\ \partial(X) &= [X] - X^\circ; & (\text{boundary of } X) \end{aligned}$$

$$\begin{aligned} \text{diam}(X) &= \sup\{\text{dist}(x, y) \mid x, y \in X\}; && \text{(diameter of } X\text{)} \\ \text{dist}(X, Y) &= \inf\{\text{dist}(x, y) \mid x \in X \wedge y \in Y\}.^{10} && \text{(distance between } X \text{ and } Y\text{)} \end{aligned}$$

Relations in  $\mathbb{R}^n$  ( $X$  and  $Y$  non-empty):

$$\begin{aligned} \text{Btw}(x, y, z) &\text{ iff } \text{dist}(y, x) + \text{dist}(x, z) = \text{dist}(y, z) && \text{(} x \text{ is between } y \text{ and } z\text{)} \\ \text{Congr}(X, Y) &\text{ iff there exists an isometry } f \text{ such that } f(X) = Y && \text{(} X \text{ is congruent to } Y\text{)} \\ \text{Conv}(X) &\text{ iff } \forall x, y, z ((x, y \in X \wedge \text{Btw}(z, x, y)) \rightarrow z \in X); && \text{(} X \text{ is a convex region)} \\ \text{Conx}(X) &\text{ iff } \forall A, B ((A^\circ \neq \emptyset \wedge B^\circ \neq \emptyset \wedge X^\circ = A^\circ \cup B^\circ) \rightarrow A^\circ \cap B^\circ \neq \emptyset); && \text{(} X \text{ is a connected region)} \\ \text{WConx}(X) &\text{ iff } \forall A, B ((A^\circ \neq \emptyset \wedge B^\circ \neq \emptyset \wedge X^\circ = A^\circ \cup B^\circ) \rightarrow [A] \cap [B] \neq \emptyset) && \text{(} X \text{ is a weakly connected region)} \\ \text{WWConx}(X) &\text{ iff } \forall A, B ((A^\circ \neq \emptyset \wedge B^\circ \neq \emptyset \wedge X^\circ = A^\circ \cup B^\circ) \rightarrow \text{dist}(A, B) = 0) && \text{(} X \text{ is a w-weakly connected region)} \end{aligned}$$

From the definitions, for all  $X \subseteq \mathbb{R}^n$ , we have  $\text{Conx}(X) \rightarrow \text{WConx}(X) \rightarrow \text{WWConx}(X)$ . The converse does not hold. However, we have  $(\text{WWConx}(X) \wedge \text{diam}(X) < +\infty) \rightarrow \text{WConx}(X)$ . These results are proven in Appendix A. Other topological and geometrical lemmas based on the notions given above become handy in proving theorems of later sections. These lemmas are collected in Appendix A as well.

### 3.2 THE THEORIES AND THEIR (NATURAL) INTERPRETATIONS

As we have seen, the natural interpretation of the non-logical primitives is crucial for the comparison. Because of this, we provide detailed notes with references to the literature and point out the cases where the information available is not satisfactory.

In this section, we list the non-logical vocabulary  $V_j$ , the domain  $D_j$ , and the (natural) interpretation  $[[\cdot]]_j$  of each mereogeometry  $\mathbf{T}_j$  we consider. Also, assume that an assignment function  $\mathfrak{S}_n$  from the set of variables to regular regions in  $\mathbb{R}^n$  has been fixed for each index  $n$ . If  $D \subseteq \wp(\mathbb{R}^n)$ , then  $[[\mathbf{R}(x, y)]]_j$  and  $[[\mathbf{R}]]_j(X, Y)$  stand for  $[[\mathbf{R}]]_j(\mathfrak{S}_n(x), \mathfrak{S}_n(y))$  where  $\mathfrak{S}_n(x) = X$  ( $X \in D_j \subseteq D$ ). Finally, since there is no danger of confusion, throughout the paper we write  $[[x]]_j$  for  $\mathfrak{S}_n(x)$  whenever the indices  $j$  and  $n$  are fixed by the context.

**T1** — Theory presented in [56] and further developed in [3, 5]:

$$\begin{aligned} V_1 &= \{\mathbf{P}, \mathbf{S}\} \text{ where} \\ \mathbf{P}(x, y) &\text{ stands for “} x \text{ is part of } y\text{” and} \\ \mathbf{S}(x) &\text{ for “} x \text{ is a sphere”}; \\ D_1 &= \{\text{non-empty regular open subsets of } \mathbb{R}^n\} \\ &= \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge [X]^\circ = X\}; \\ [[\mathbf{P}(x, y)]]_1 &= X \subseteq Y; \\ [[\mathbf{S}(x)]]_1 &= \exists c \in \mathbb{R}^n, r \in \mathbb{R}^+ (X = \text{ball}(c, r)). \end{aligned}$$

**T2** — Theory presented in [8]:<sup>11</sup>

$$\begin{aligned} V_2 &= \{\mathbf{P}, \mathbf{SR}, \mathbf{CG}\} \text{ where} \\ \mathbf{P}(x, y) &\text{ stands for “} x \text{ is part of } y\text{”,} \end{aligned}$$

<sup>10</sup> Sometimes we write  $\text{dist}(x, Y)$  for  $\text{dist}(\{x\}, Y)$ . Analogously for  $\text{dist}(X, y)$ .

<sup>11</sup> This theory is restricted to  $\mathbb{R}^3$  since it was developed for the description of physical objects. One can easily generalize it to  $\mathbb{R}^n$  changing the axioms on the space dimension.

$\text{SR}(x)$  for “ $x$  is a simple region” (or “ $x$  is connected”), and  
 $\text{CG}(x, y)$  for “ $x$  is congruent to  $y$ ”;  
 $D_2 = \{\text{non-empty regular open subsets of } \mathbb{R}^n \text{ with finite diameter}\}$   
 $= \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge [X]^\circ = X \wedge \text{diam}(X) < +\infty\}$ ;  
 $\llbracket \text{P}(x, y) \rrbracket_2 = X \subseteq Y$ ;  
 $\llbracket \text{SR}(x) \rrbracket_2 = \text{Conx}(X)$ ;  
 $\llbracket \text{CG}(x, y) \rrbracket_2 = \text{Congr}(X, Y)$ .

**T3** — Theory given in [39]:<sup>12</sup>

$V_3 = \{\text{P, Conj}\}$  where  
 $\text{P}(x, y)$  stands for “ $x$  is part of  $y$ ” and  
 $\text{Conj}(x, y, z, w)$  for “ $x, y$  and  $z, w$  are conjugate”;  
 $D_3 = \{\text{non-empty regular closed connected subsets of } \mathbb{R}^n\}$   
 $= \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge [X]^\circ = X \wedge \text{Conx}(X)\}$ ;  
 $\llbracket \text{P}(x, y) \rrbracket_3 = X \subseteq Y$ ;  
 $\llbracket \text{Conj}(x, y, x', y') \rrbracket_3 = \exists x, y, x', y' (x \in X \wedge y \in Y \wedge x' \in X' \wedge y' \in Y' \wedge \text{dist}(x, y) = \text{dist}(x', y'))$ .

**T4** — Theory introduced in [20] and further developed in [21]:<sup>13</sup>

$V_4 = \{\text{CCon}\}$  where  
 $\text{CCon}(x, y, z)$  stands for “ $x$  can connect both  $y$  and  $z$ ”;  
 $D_4 = \{\text{non-empty regular closed connected subset of } \mathbb{R}^n \text{ with finite diameter}\}$   
 $= \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge [X]^\circ = X \wedge \text{Conx}(X) \wedge \text{diam}(X) < +\infty\}$ ;  
 $\llbracket \text{CCon}(z, x, y) \rrbracket_4 = \text{dist}(X, Y) \leq \text{diam}(Z)$ .

**T5** — Theory first introduced in [58] and further developed in [2]:<sup>14</sup>

$V_5 = \{\text{C, Closer}\}$  where  
 $\text{C}(x, y)$  stands for “ $x$  is connected to  $y$ ” and  
 $\text{Closer}(x, y, z)$  for “ $x$  is closer to  $y$  than to  $z$ ”;  
 $D_5 = \{\text{non-empty regular subsets of } \mathbb{R}^n\}$ <sup>15</sup>  
 $= \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge ([X] = [X]^\circ) \wedge X^\circ = [X]^\circ\}$ ;  
 $\llbracket \text{C}(x, y) \rrbracket_5 = X \cap Y \neq \emptyset$ ;  
 $\llbracket \text{Closer}(z, x, y) \rrbracket_5 = \text{dist}(Z, X) < \text{dist}(Z, Y)$ .

**T6** — Theory given in [13], [15], and [12]:

<sup>12</sup> Nicod provides an informal description of  $\text{Conj}$ : “[D]eux volumes  $AA'$ ,  $BB'$  sont *conjugués* s'il existe un point de  $A$  et un point de  $A'$ , un point de  $B$  et un point de  $B'$ , séparés par la même distance” ([39], pp.27-28). He also characterizes the domain, see [39], p.27.

<sup>13</sup> De Laguna provides an informal description of  $\text{CCon}$ : “[T]o say that  $C$  can connect  $A$  and  $B$  would be understood to mean that we could, if we wished, put  $C$  in simultaneous contact with  $A$  and  $B$ ” ([20], p.450), i.e., considering a domain of closed regions,  $\llbracket \text{CCon}(x, y, z) \rrbracket = \exists x, x', y, z (x, x' \in X \wedge y \in Y \wedge z \in Z \wedge \text{dist}(x, x') = \text{dist}(y, z))$ . In [21] a different interpretation is given:  $\llbracket \text{CCon}(z, x, y) \rrbracket_4 = \text{dist}(X, Y) \leq \text{diam}(Z)$ . This interpretation is equivalent to the one of De Laguna in  $D_4$ , but, as shown in section 5.1.1, in the domains containing non-connected regions it does not satisfy all the axioms provided by De Laguna. Since we are working in domains containing non-connected regions as well, we consider the weaker one.

<sup>14</sup> No intended interpretation for the relation  $\text{Closer}$  is provided. The interpretation we propose seems self-evident and satisfies all the axioms. As far as we can tell, it is faithful to this approach.

<sup>15</sup> [58] considers only convex and bounded regions.

$V_6 = \{\mathbf{C}, \text{ConvH}\}$  where

$\mathbf{C}(x, y)$  stands for “ $x$  is connected to  $y$ ” and

$\text{ConvH}(x, y)$  for “ $x$  is the convex hull of  $y$ ”;

$D_6 = \{\text{non-empty regular open subsets of } \mathbb{R}^n\}$

$= \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge [X]^\circ = X\}$ ;

$\llbracket \mathbf{C}(x, y) \rrbracket_6 = [X] \cap [Y] \neq \emptyset$ ;

$\llbracket \text{ConvH}(x, y) \rrbracket_6 = \text{Conv}(X) \wedge Y \subseteq X \wedge \neg \exists Z(\text{Conv}(Z) \wedge Y \subseteq Z \wedge Z \subset X)$ .

#### 4 ENVIRONMENT STRUCTURES

Having listed all the systems and their (natural) interpretations, the next issue is the definition of the environment structures for the comparison. The key step in this part of our method is the choice of the structure domains.

As we have seen, all mereotopologies in section 3 consider non-empty regular regions and do not refer to lower-dimension objects like points or boundaries. Looking at the descriptions, we see that all the domains, with the only exception of  $D_5$ , contain regular regions only and these are either all open or all closed (perhaps with additional constraints). In particular,  $D_1 = D_6 = \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge X = [X]^\circ\}$  is the set of regular open regions. Let us call this domain  $D_O$ . From the topological point of view,  $D_O$  is the counterpart of  $D_C = \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge X = [X^\circ]\}$ , the set of regular closed regions, since there exists a bijection  $f: D_O \rightarrow D_C$  given by  $f(X) = [X]$  such that  $f^{-1}(X) = X^\circ$ ,  $f^{-1}(f(X)) = X$ , and  $f(f^{-1}(X)) = X$ .

Domain  $D_5$  is the union of  $D_O$  and  $D_C$ . In  $D_5$ , set-inclusion ( $\subseteq$ ) is non-extensional: if  $X$  is a regular closed region in  $D_5$ , then  $X^\circ \in D_5$  and  $X^\circ \subset X$  but there is nothing in  $D_5$  that makes the difference between these elements. This is a major difference between **T5** and the other systems, and it has far reaching consequences already at the mereotopological level [17]. In this paper we concentrate on  $\subseteq$ -extensional systems, thus we leave out the domain  $D_5$ . Indeed, set-inclusion is extensional in all the other theories of section 3 and it is used extensively to prove the results of section 5. This does not mean that we dismiss **T5** altogether. Instead, we take the interpretations of **C** and **Closer** provided by the authors of **T5** and apply these in the other domains. Admittedly, this covers only one direction in the comparison of **T5** with the other systems, but the analysis of theories **T1–T6** in the natural domain of **T5** is quite complex and we leave this subject for future work.

Focusing on the relationship between  $D_O$  and  $D_C$ , one wonders if it is possible to reduce the comparison to domains containing open regions (or closed regions) only. Let us start with a specific example. Two of our theories are interpreted in the class of closed regions, namely **T3** (primitives **P** and **Conj**) and **T4** (primitive **CCon**). From Lemma L.5 (see Appendix A) we see that the interpretation of **P**, i.e.  $\subseteq$ , is independent from the open/closed distinction. The interpretation of **CCon** behaves analogously since from Lemmas L.8 and L.9 we have  $\text{dist}(X, Y) = \text{dist}([X], [Y]) = \text{dist}(X^\circ, Y^\circ)$  and  $\text{diam}(X) = \text{diam}([X]) = \text{diam}(X^\circ)$ . The case of **Conj** is different. The natural interpretation (in  $D_C$ ) is given by the formula

$$(4.1) \quad \exists p, q, p', q' (p \in X \wedge q \in Y \wedge p' \in X' \wedge q' \in Y' \wedge \text{dist}(p, q) = \text{dist}(p', q'))$$

If we want to make justice of this interpretation in the domain  $D_O$ , we should consider the topological closures of the variable values, that is, the following formula<sup>16</sup>

$$(4.2) \quad \exists p, q, p', q' (p \in [X] \wedge q \in [Y] \wedge p' \in [X'] \wedge q' \in [Y'] \wedge \text{dist}(p, q) = \text{dist}(p', q'))$$

However, formulas (4.1) and (4.2) are equivalent in  $D_C$  only. For a counterexample in  $D_O$ , let  $n=1$  and take  $X = (0, 1)$ ,  $Y = (4, 5)$ ,  $X' = (1, 2)$ , and  $Y' = (3, 4)$ . Thus, in both  $D_O$  and  $D_C$ , formula (4.2) coincides with the (informal) interpretation provided by the authors while (4.1) is specialized to  $D_C$  since it takes advantage of the properties of that domain. In particular, taking (4.2) to be the interpretation for **Conj**, our comparison can be carried out in both the domains  $D_O$  and  $D_C$ . This example suggests a way to reliably restrict our analysis to a subclass of regular regions, say  $D_O$ , by allowing us to ‘transfer’ the results to the other,  $D_C$ . Indeed, taking the closure of the values given by the assignment function, that is, substituting  $[X]$  for any occurrence of  $X$  in the interpretation of a formula, we can interpret in  $D_O$  also those primitives that are defined on the domain of regular closed regions.

Finally, since some primitives seem to occur in several systems, we need to verify that the corresponding interpretations are compatible. If this is not the case, these have to be taken as distinct primitives. In our case, two primitives occur in several theories: **P** is included in **T1**, **T2**, and **T3**; **C** is included in **T5** and **T6**. **P** has the same interpretation in all the theories, thus we can identify the primitive **P** in all these theories. Regarding **C**, its interpretation differs in the two domains  $D_5$  and  $D_6$ . Since the interpretation in **T5** is based on a domain that we do not consider in this paper, for this primitive we embrace the interpretation provided by **T6**. This choice seems compatible with **T5** since, in the case of open regions, **T5** interprets **C** as the standard overlap relation which is less expressive than any connection relation while, in the case of closed regions, **T5** and **T6** agree in the interpretation of **C**. Furthermore, we notice that there exists a definition of **C** in terms of **Closer** (the only other primitive of **T5**) that is compatible with the interpretation of **C** we have chosen. That is, in either case the relation **C** is definable through **Closer** in  $D_O$  or  $D_C$ . This definition is given in section 5.2.

Putting things together, we end up with eight distinct environment structures:  $\Phi_\alpha - \Phi_\theta$ . These have fixed vocabulary  $V = \{\mathbf{C}, \mathbf{CCon}, \mathbf{CG}, \mathbf{Closer}, \mathbf{Conj}, \mathbf{ConvH}, \mathbf{P}, \mathbf{S}, \mathbf{SR}\}$  and fixed interpretation functions (see Table 1) but different domains (see Table 2). Recall that each theory is associated with a specific domain, i.e., the domain of its natural model. As a consequence, with the exception of theory **T5**, we associate each theory with the structure isolated by its natural domain and the interpretation functions given at the end of section 4 and call this the *natural environment structure* for that theory (see Table 2).

**Definition 5.1.1** — *Let **T** be a theory and  $D$  the domain of its natural model. The natural environment structure of **T** is the environment structure among the  $\Phi_\alpha - \Phi_\theta$  that has domain  $D$ .*

$\llbracket \mathbf{C}(x, y) \rrbracket = [X] \cap [Y] \neq \emptyset$
$\llbracket \mathbf{CCon}(z, x, y) \rrbracket = \text{dist}(X, Y) \leq \text{diam}(Z)$
$\llbracket \mathbf{CG}(x, y) \rrbracket = \text{Congr}(X, Y)$
$\llbracket \mathbf{Closer}(z, x, y) \rrbracket = \text{dist}(Z, X) < \text{dist}(Z, Y)$

<sup>16</sup> Recall that, in a formula, the interpretation of the variables is restricted to the domain  $D_O$ . However, the formula itself may refer to regions outside  $D_O$ .

$\llbracket \text{Conj}(x, y, x', y') \rrbracket = \exists x, y, x', y' (x \in [X] \wedge y \in [Y] \wedge x' \in [X'] \wedge y' \in [Y'] \wedge \text{dist}(x, y) = \text{dist}(x', y'))$
$\llbracket \text{ConvH}(x, y) \rrbracket = \text{Conv}(X) \wedge Y \subseteq X \wedge \neg \exists Z (\text{Conv}(Z) \wedge Y \subseteq Z \wedge Z \subseteq X)$
$\llbracket \text{P}(x, y) \rrbracket = X \subseteq Y$
$\llbracket \text{S}(x) \rrbracket = \exists c, r (X^\circ = \text{ball}(c, r))$ with $r \in \mathbb{R}^+$
$\llbracket \text{SR}(x) \rrbracket = \text{Conv}(X)$

**Table 1.** Interpretation of the vocabulary on non-empty regular regions of  $\mathbb{R}^n$ .<sup>17</sup>

STRUCT.	DOMAIN	DOMAIN DESCRIPTION <i>non-empty regular regions and...</i>	NATURAL ENV.
$\Phi_\alpha$	$D_\alpha = D_O = \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge [X]^\circ = X\}$	<i>open</i>	<b>T1, T6</b>
$\Phi_\beta$	$D_\beta = \{X \in D_O \mid \text{diam}(X) < +\infty\}$	<i>open and finite</i>	<b>T2</b>
$\Phi_\gamma$	$D_\gamma = \{X \in D_O \mid \text{Conv}(X)\}$	<i>open and connected</i>	
$\Phi_\delta$	$D_\delta = \{X \in D_O \mid \text{Conv}(X) \wedge \text{diam}(X) < +\infty\}$	<i>open, finite and connected</i>	
$\Phi_\epsilon$	$D_\epsilon = D_C = \{X \subseteq \mathbb{R}^n \mid X \neq \emptyset \wedge [X]^\circ = X\}$	<i>closed</i>	<b>T3</b>
$\Phi_\zeta$	$D_\zeta = \{X \in D_C \mid \text{diam}(X) < +\infty\}$	<i>closed and finite</i>	
$\Phi_\eta$	$D_\eta = \{X \in D_C \mid \text{Conv}(X)\}$	<i>closed and connected</i>	
$\Phi_\theta$	$D_\theta = \{X \in D_C \mid \text{Conv}(X) \wedge \text{diam}(X) < +\infty\}$	<i>closed, finite and connected</i>	<b>T4</b>

**Table 2.** Structures and their domains.

From section 2, given two theories and a domain, say **T1**, **T4** and  $D_\alpha$ , the theories are said to be  $\Phi_\alpha$ -equivalent if and only if the relation  $\text{dist}(X, Y) \leq \text{diam}(Z)$  can be defined in the structure  $\Phi_\alpha(\mathbf{T1}) = \langle D_\alpha, X \subseteq Y, \exists c, r (X^\circ = \text{ball}(c, r))$  with  $r \in \mathbb{R}^+$ , and the relations  $X \subseteq Y$  and  $\exists c, r (X^\circ = \text{ball}(c, r))$  can be defined in the structure  $\Phi_\alpha(\mathbf{T4}) = \langle D_\alpha, \text{dist}(X, Y) \leq \text{diam}(Z) \rangle$ . In other terms, both the structures  $\Phi_\alpha(\mathbf{T1})$  and  $\Phi_\alpha(\mathbf{T4})$  can be *definitionally expanded* to the structure  $\langle D_\alpha, \text{dist}(X, Y) \leq \text{diam}(Z), X \subseteq Y, \exists c, r (X^\circ = \text{ball}(c, r))$  with  $r \in \mathbb{R}^+$ , see [31].

Before moving to next section, we add a couple of words to motivate our choice of **CCon**'s interpretation. The **CCon** primitive is introduced in a domain of connected regions and there are different ways to generalize it to the more comprehensive domain we consider in this paper. At first sight, when working in the domain of all open regular regions, one might want to impose that in **CCon**( $x, y, z$ ) variable  $x$  must range over connected regions only. This constraint would capture the intuition that whenever a region “can connect” two given regions, then it “can connect” any two regions that are at closer distance. However, this constraint seems too strong with respect to the underlying intuition since it is also quite intuitive to say that (in  $\mathbb{R}^1$ ) region  $X = (0,1) \cup (2,3)$  “can connect”  $Y = (10,12)$  and  $Z = (13,15)$ . From a broader perspective, the problem is to understand which properties of the primitive that are guaranteed by the peculiarity of the original domain, should be explicitly enforced in the more general interpretation. Our approach in these cases is to adopt the interpretation that makes the primitive weaker. Such a choice allows us to better analyze the import of the primitive. For the sake of completeness, note that one could informally interpret **CCon**( $x, y, z$ ) as “there are two points of  $x$  whose distance is equal to the distance between a point of  $y$  and a point of  $z$ .” This interpretation has been discharged for the simple reason that it would make **CCon** a subcase of **Conj**.

<sup>17</sup> In this case, we indicate the interpretation function with the double brackets  $\llbracket \cdot \rrbracket$  without indices.

## 5 TRANSLATIONS BETWEEN THEORIES

In the previous sections we have prepared the elements for the formal comparison. The mereotopometries that we consider have been presented in section 3 and the environment structures have been chosen and motivated in section 4. Now, we enter into the actual comparison giving the formal results. In this section we collect the theorems, while their proofs, which are sometimes long and involved, are presented in the appendices.

Since in the previous section we have shown how to reduce the comparison to structures with domain in  $D_O$  only, we abuse the above notation by using  $\Phi_\alpha$ - $\Phi_\delta$  as natural environment structures even for theories whose natural domain  $D$  is contained in  $D_C$ .

### 5.1 VERIFYING THE GIVEN EXPLICIT DEFINITIONS

First we prove that the explicit definitions, provided by each theory and of interest for the comparison, are satisfied in the natural environment structure for that theory. We will see that there is one exception: the definition proposed in **T1** to capture the relation **CCon** yields an interpretation function that does not satisfy the axiomatization of the primitive **CCon** given in **T4**. We will show how to modify such a definition to capture the correct meaning of the primitive. Regarding **T5**, we do not take into account the definitions it provides because they are conceived for the domain  $D_5$ .

First, we consider the (derived) interpretations of those mereotopological notions that receive a common definition in the theories **T1–T6**. These notions are extensively used in the rest of the paper.

**Proposition 5.1.1** — *In all the structures  $\Phi_{\alpha-\delta}$ , the following holds*

- (DP) Let  $P^*(x, y) =_{\text{def}} \forall w(C(x, w) \rightarrow C(y, w))$ , then  
 $\llbracket P^*(x, y) \rrbracket_{\alpha-\delta} = X \subseteq Y$ ,<sup>18</sup> (*x is a part of y*)
- (DPP) Let  $PP(x, y) =_{\text{def}} P(x, y) \wedge \neg P(y, x)$ , then  
 $\llbracket PP(x, y) \rrbracket_{\alpha-\delta} = X \subset Y$ ; (*x is a proper part of y*)
- (DO) Let  $O(x, y) =_{\text{def}} \exists z(P(z, x) \wedge P(z, y))$ , then  
 $\llbracket O(x, y) \rrbracket_{\alpha-\delta} = X \cap Y \neq \emptyset$ ; (*x and y overlap*)
- (DPO) Let  $PO(x, y) =_{\text{def}} O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$ , then  
 $\llbracket PO(x, y) \rrbracket_{\alpha-\delta} = X \cap Y \neq \emptyset \wedge \neg X \subseteq Y \wedge \neg Y \subseteq X$ ; (*x and y partially overlap*)
- (D+) Let  $SUM(z, x, y) =_{\text{def}} \forall w(O(w, z) \leftrightarrow (O(w, x) \vee O(w, y)))$ , then  
 $\llbracket SUM(z, x, y) \rrbracket_{\alpha-\delta} = (Z = [X \cup Y]^o)$ ; (*z is the sum of x and y*)
- (D−) Let  $DIF(z, x, y) =_{\text{def}} \forall w(P(w, z) \leftrightarrow (P(w, x) \wedge \neg O(w, y)))$ , then  
 $\llbracket DIF(z, x, y) \rrbracket_{\alpha-\delta} = (Z = X - [Y])$ ; (*z is x minus y*)
- (DEC) Let  $EC(x, y) =_{\text{def}} C(x, y) \wedge \neg O(x, y)$ , then  
 $\llbracket EC(x, y) \rrbracket_{\alpha-\delta} = [X] \cap [Y] \neq \emptyset \wedge X \cap Y = \emptyset$ . (*x and y are externally connected*)
- (DIP) Let  $IP(x, y) =_{\text{def}} P(x, y) \wedge \forall z(C(z, x) \rightarrow O(z, y))$ , then  
 $\llbracket IP(x, y) \rrbracket_{\alpha-\delta} = [X] \subseteq Y$ ; (*x is an interior part of y*)

<sup>18</sup> Note that the interpretation of  $P^*$  is identical to the interpretation fixed for  $P$ . For this reason, in the following we will identify  $P^*$  with  $P$ .

- (DTPP) Let  $\text{TPP}(x, y) =_{\text{def}} \text{PP}(x, y) \wedge \exists z(\text{EC}(z, x) \wedge \text{EC}(z, y))$ , then  
 $\llbracket \text{TPP}(x, y) \rrbracket_{\alpha-\delta} = X \subset Y \wedge \partial(X) \cap \partial(Y) \neq \emptyset$ ; *(x is an tangential proper part of y)*
- (DSC) Let  $\text{SC}(x) =_{\text{def}} \forall y, z(\text{SUM}(x, y, z) \rightarrow \text{C}(y, z))$ , then  
 $\llbracket \text{SC}(x) \rrbracket_{\alpha-\delta} = \text{WConx}(X)$ ; *(x is weakly connected)*

*Proof*— See Appendix B.1. ♦

We are now ready to analyze the definitions provided in some theories, namely **T1**, **T2**, **T4**, and **T6**. The goal here is to ensure that these definitions capture correctly the intended notions, to provide counterexamples where they do not, and to propose a corrected version when needed. To distinguish the vocabulary of the environment structures in section 4 from the relation symbols within a theory, we label those in the latter group with the index of the theory where they occur. For instance,  $\text{C}_1(x, y)$  is the connection relationship defined in theory **T1** while  $\text{C}(x, y)$  is the connection relationship with interpretation as in Table 1.

### 5.1.1 DEFINITIONS IN T1

*Primitives of T1*: P, S.

The explicit definitions provided in this theory involve two relationships that we do not discuss directly. The first, **CNC**, is the relationship that holds between two concentric spheres and is introduced and defined by Tarski in [56]. More precisely, in this paper Tarski proves  $\llbracket \text{CNC}(x, y) \rrbracket = (X = \text{ball}(c, r) \wedge Y = \text{ball}(c, r'))$  for any  $c, r$ , and  $r' (r, r' > 0)$ . We do not repeat the argument and refer the reader to that paper on this topic. **CNC** is adopted in **T1** without changes and it is used to define the connection relation as shown in (DC1) below. The other is **CG**, which is needed to capture **CCon**. The definition of **CG** given in the language of **T1** is quite complex. In section 5.3.1 we provide an improved definition of **CG** (within theory **T1**) that works in all the domains. There we also prove that its interpretation corresponds to that of section 4. For the time being, we show that the existing definition of **CCon** given in [5] must be corrected.

*Explicit definitions furnished in T1*:

- (DC1)  $\text{C}_1(x, y) =_{\text{def}} \exists z(\text{S}(z) \wedge \forall z'(\text{CNC}(z', z) \rightarrow (\text{O}(z', x) \wedge \text{O}(z', y))))$ ;
- (DSR1)  $\text{SR}_1(x) =_{\text{def}} \forall y, z(\text{SUM}(x, y, z) \rightarrow \exists s(\text{S}(s) \wedge \text{O}(s, y) \wedge \text{O}(s, z) \wedge \text{P}(s, x)))$ ;
- (DCC1)  $\text{CCon}_1(z, x, y) =_{\text{def}} \exists z'(\text{CG}(z', z) \wedge \text{C}_1(z', x) \wedge \text{C}_1(z', y))$ .

**Proposition 5.1.2** —  $\llbracket \text{C}_1(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{C}(x, y) \rrbracket$  and  $\llbracket \text{SR}_1(x) \rrbracket_{\alpha-\delta} = \llbracket \text{SR}(x) \rrbracket$ .

*Proof*— See Appendix B.2. ♦

**Proposition 5.1.3** — Let  $\llbracket \text{CG}(x, y) \rrbracket = \text{Congr}(X, Y)$ , then:

$$\llbracket \text{CCon}_1(z, x, y) \rrbracket_{\alpha} = \exists z, z', x, y(z, z' \in [Z] \wedge x \in [X] \wedge y \in [Y] \wedge \text{dist}(x, y) = \text{dist}(z, z'))$$

*Proof*. See Appendix B.3. ♦

In  $D_{\alpha}$  (i.e. the natural domain of **T1**) the interpretation of  $\text{CCon}_1$  is not equivalent to the interpretation of **CCon** given in section 4. For example, let  $n = 1$  and take  $X = (2, 3)$ ,  $Y = (5, 6)$ , and  $Z =$

$(0, 1) \cup (7, 8)$ . Then,  $\text{dist}(X, Y) \leq \text{diam}(Z)$  but  $\neg \exists z, z', x, y (z, z' \in Z \wedge x \in X \wedge y \in Y \wedge \text{dist}(x, y) = \text{dist}(z, z'))$ . In addition, while the interpretation in section 4 satisfies all the axioms given by De Laguna on the *can-connect* primitive in all the structures, the interpretation of  $\text{CCon}_1$  does not satisfy (in  $\Phi_\alpha$ ) the following De Laguna axiom:

$$\exists x, y (\text{CCon}(a, x, y) \wedge \neg \text{CCon}(b, x, y)) \rightarrow \neg \exists z, v (\text{CCon}(b, z, v) \wedge \neg \text{CCon}(a, z, v))$$

A counterexample in  $D_\alpha$  (for  $n = 1$ ) is obtained by taking  $A = (3, 5)$ ,  $B = (0, 1) \cup (7, 8)$ ,  $X = (2, 3)$ ,  $Y = (5, 6)$  and  $Z = (-1, 0)$ ,  $V = (8, 9)$ .

We conclude that (DCC1) does not capture the De Laguna's *can-connect* primitive and therefore, in our conceptual comparison, we will not make use of this definition.

### 5.1.2 DEFINITIONS IN T2

*Primitives of T2:* P, SR, CG.

The definition (DC2), given below, uses relationship CNC. This has been discussed in section 5.1.1 and is adopted in T2 without changes with respect to Tarski's work. Of course, the correctness of CNC in this theory depends on the correctness of (DS2) which defines what counts as a sphere in this theory.

*Explicit definitions furnished in T2:*

$$(DS2) \quad S_2^*(x) =_{\text{def}} \text{SR}(x) \wedge \forall y, z ((\text{CG}(x, y) \wedge \text{PO}(x, y) \wedge \text{DIF}(z, x, y)) \rightarrow \text{SR}(z));$$

$$(DC2) \quad C_2(x, y) =_{\text{def}} \exists z (S_2^*(z) \wedge \forall z' (\text{CNC}(z', z) \rightarrow (\text{O}(z', x) \wedge \text{O}(z', y)))).$$

**Proposition 5.1.4** —  $\llbracket S_2^*(x) \rrbracket_\beta = \llbracket S(x) \rrbracket$  ( $\Phi_\beta$  is the natural environment structure of T2) and  $\llbracket S_2^*(x) \rrbracket_{\alpha, \gamma, \delta} \neq \llbracket S(x) \rrbracket$ .

*Proof.* See Appendix B.4. ♦

**Proposition 5.1.5** —  $\llbracket C_2(x, y) \rrbracket = \llbracket C(x, y) \rrbracket$  provided  $\llbracket S_2^*(x) \rrbracket = \llbracket S(x) \rrbracket$ .

*Proof.* It follows from the proof of Proposition 5.1.2. ♦

[4] proposes a definition of S based on P and CG together with an attempt to provide semantic equivalence in the domain of open regular regions. We are not going to consider this definition in our comparison because, unfortunately, it fails to capture the notion of sphere. The interested reader can easily verify that the definition in that paper does not rule out non-spherical regions like Reuleaux polytopes<sup>19</sup>.

### 5.1.3 DEFINITIONS IN T4

*Primitive of T4:* CCon.

*Explicit definitions furnished in T4:*

$$(DC4^*) \quad C_4^*(x, y) =_{\text{def}} \forall z (\text{CCon}(z, x, y));$$

$$(DP4^*) \quad P_4^*(x, y) =_{\text{def}} \forall z (\text{C}_4^*(z, x) \rightarrow \text{C}_4^*(z, y));$$

$$(DP4^+) \quad P_4^+(x, y) =_{\text{def}} \forall z, w (\text{CCon}(w, z, x) \rightarrow \text{CCon}(w, z, y));$$

<sup>19</sup> A Reuleaux polytope in  $\mathbb{R}^2$  is the region obtained by intersecting the three discs centered at the vertices of an equilateral triangle with radius (of length) equal to (the length of) the side of the triangle itself.

(DC14)  $\text{Closer}_4(z, x, y) =_{\text{def}} \exists a(\text{CCon}(a, z, x) \wedge \neg \text{CCon}(a, z, y))$ .

**Proposition 5.1.6** —  $\llbracket \text{C}_4^*(x, y) \rrbracket_{\beta, \delta} = \llbracket \text{C}(x, y) \rrbracket$  ( $\Phi_\delta$  is the open counterpart of  $\Phi_0$ , i.e. of the natural environment structure of **T4**) and  $\llbracket \text{C}_4^*(x, y) \rrbracket_{\alpha, \gamma} \neq \llbracket \text{C}(x, y) \rrbracket$ .

*Proof* — See Appendix B.5. ♦

**Proposition 5.1.7** —  $\llbracket \text{P}_4^*(x, y) \rrbracket_{\alpha, \delta} = \llbracket \text{P}(x, y) \rrbracket$ ,  $\llbracket \text{P}_4^+(x, y) \rrbracket_{\alpha, \delta} = \llbracket \text{P}(x, y) \rrbracket$ , and  $\llbracket \text{Closer}_4(z, x, y) \rrbracket_{\alpha, \delta} = \llbracket \text{Closer}(z, x, y) \rrbracket$ .

*Proof* — See Appendix B.6. ♦

#### 5.1.4 DEFINITIONS IN T6

*Primitives of T6:* C, ConvH.

*Explicit definitions furnished in T6:*<sup>20</sup>

(DP6)  $\text{P}_6(x, y) =_{\text{def}} \forall z(\text{C}(z, x) \rightarrow \text{C}(z, y))$ ;

(DCM6)  $\text{Compl}_6(y, x) =_{\text{def}} \forall z(\text{C}(z, y) \leftrightarrow \neg \text{IP}(z, x))$ ;<sup>21</sup>

(DSR6)  $\text{SR}_6(x) =_{\text{def}} \forall y, z, w((\text{SUM}(x, y, z) \wedge \text{Compl}_6(w, x)) \rightarrow \exists v(\text{SC}(v) \wedge \text{O}(v, y) \wedge \text{O}(v, z) \wedge \neg \text{C}(v, w)))$ .<sup>22</sup>

**Proposition 5.1.8** —  $\llbracket \text{P}_6(x, y) \rrbracket_{\alpha, \delta} = \llbracket \text{P}(x, y) \rrbracket$ , and  $\llbracket \text{SR}_6(x) \rrbracket_{\alpha, \delta} = \llbracket \text{SR}(x) \rrbracket$ .

*Proof* — See Appendix B.7. ♦

## 5.2 DISPENSABLE PRIMITIVES

Using extensively the definitions and results of the previous sections, we now investigate if some primitives of a theory **Ti** can be defined (in all  $\Phi_{\alpha, \delta}$ ) on the basis of the other primitives of the same theory **Ti**. This ‘internal reduction’ points out some redundancies and reduced that steps needed to compare the theories.

**Proposition 5.2.1** — In **T2** we can use (DSR6) to define the relation SR.

*Proof* — Directly from Proposition 5.1.8 and the fact that all the predicates used in (DSR6) are definable in **T2** with the same interpretation. ♦

In **T3**, on the basis of the primitive Conj we can define the parthood relation:

(DC3\*)  $\text{C}_3^*(x, y) =_{\text{def}} \forall z(\text{Conj}(z, z, x, y))$ ;

(DP3)  $\text{P}_3(x, y) =_{\text{def}} \forall z(\text{C}_3^*(z, x) \rightarrow \text{C}_3^*(z, y))$ .

<sup>20</sup> The predicates O, IP, SUM, and SC in the definitions (DCM6) and (DSR6) are defined using  $\text{P}_6$ . Because of Propositions 5.1.1, and 5.1.8, their definitions coincide with the original ones. For this reason, we do not introduce new symbols for these predicates.

<sup>21</sup> In the original papers, the complement is introduced as a primitive function. Here we adopt the standard formulation of the complement as a relation.

<sup>22</sup> This predicate is called ‘manifold’ in [13].

**Proposition 5.2.2** —  $\llbracket P_3(x, y) \rrbracket_{\alpha-\delta} = \llbracket P(x, y) \rrbracket$ .

*Proof* — See Appendix B.8. ♦

In **T5**, on the basis of the primitive **Closer** we can define the connection relation:

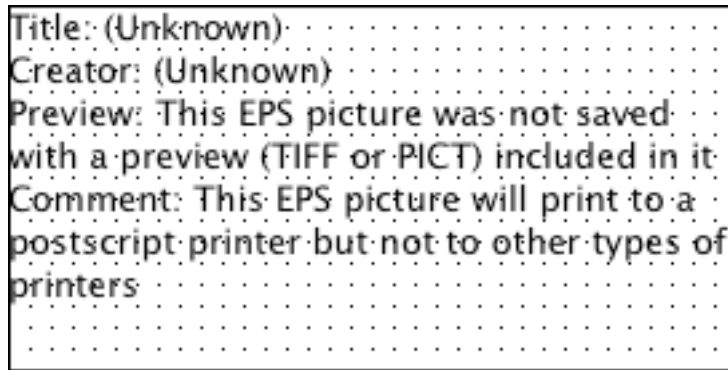
- (DC5\*)  $C_5^*(x, y) =_{\text{def}} \neg \exists z (\text{Closer}(x, z, y))$ ;  
 (DP5)  $P_5(x, y) =_{\text{def}} \forall z (C_5^*(z, x) \rightarrow C_5^*(z, y))$ ;  
 (DFD5)  $FD_5(x) =_{\text{def}} \exists z (\forall x', x'' ((P_5(x', x) \wedge P_5(x'', x)) \rightarrow \text{Closer}(x', x'', z)))$ ; (*x has finite diameter*)  
 (DC5)  $C_5(x, y) =_{\text{def}} \exists z, w (FD_5(z) \wedge FD_5(w) \wedge P_5(z, x) \wedge P_5(w, y) \wedge C_5^*(z, w))$ .

**Proposition 5.2.3** —  $\llbracket C_5(x, y) \rrbracket_{\alpha-\delta} = \llbracket C(x, y) \rrbracket$ .

*Proof* — See Appendix B.9. ♦

### 5.3 LINKING VIA EXPLICIT DEFINITIONS

In this section, we show how the mereogeometries **T1–T6** are related in the structures  $\Phi_{\alpha-\theta}$ . The connections are illustrated in Figure 1, which shows our main result. At the end of this section, we will be able to conclude that these mereogeometries, with the exception of **T6**, are actually  $\Phi_{\alpha-\theta}$ -equivalent.



**Figure 1.** Definitional links between mereogeometries (for each theory, the primitives are indicated).

Here ‘**Ti** → **Tj**’ means ‘theory **Tj** is a  $\Phi_{\alpha-\theta}$ -subtheory of **Ti**’, i.e., in  $\Phi_{\alpha-\theta}$ , all the primitives in **Tj** can be defined on the basis of the primitives of **Ti** (the labels indicates the section in which the proof is given).

#### 5.3.1 FROM T1 TO T2

By Proposition 5.1.2, in **T1**, (DC1) defines **C** in all the structures, therefore we can use all the relations defined in Proposition 5.1.1. We use the additional relations **ID**( $z, x, y$ ) (*x and y are internally diametrical with respect to z*) and **CNC**( $x, y$ ) (*x is concentric with y*) which were introduced by Tarski in [56]. As done before, we report only the interpretations. A full description and the related proof of correctness can be found in [56].

$$\llbracket \text{ID}(z, x, y) \rrbracket_{\alpha-\delta} = \text{IntD}(Z, X, Y)$$

(the centers of  $z, x, y$  are aligned and  $z$  is the minimum sphere containing  $x, y$ )

$$\llbracket \text{CNC}(x, y) \rrbracket_{\alpha-\delta} = \exists c, r_1, r_2 (X_1 = \text{ball}(c, r_1) \wedge X_2 = \text{ball}(c, r_2));$$

Using all these relations, we can define when two regions are congruent

$$\begin{aligned} \text{CG}_1(x, y) =_{\text{def}} & \forall z(\Sigma\text{SS}(z) \rightarrow \exists z'(\Sigma\text{CG}(z, z') \wedge \forall s, s'((\text{MSP}(s, z) \wedge \text{MSP}(s', z') \wedge \text{SCG}(s, s')) \rightarrow \\ & ((\text{P}(s, x) \leftrightarrow \text{P}(s', y)) \wedge (\text{P}(s, y) \leftrightarrow \text{P}(s', x)) \wedge (\text{PO}(s, x) \leftrightarrow \text{PO}(s', y)) \wedge (\text{PO}(s, y) \leftrightarrow \text{PO}(s', x)))))) \\ & \text{(} x \text{ and } y \text{ are congruent regions)} \end{aligned}$$

where

$$\begin{aligned} \text{SCG}(x, y) =_{\text{def}} & \text{S}(x) \wedge \text{S}(y) \wedge (x = y \vee \exists z, w(\text{CNC}(z, w) \wedge \text{EC}(z, x) \wedge \text{EC}(z, y) \wedge \text{TPP}(x, w) \wedge \text{TPP}(y, w))) \\ & \text{(} x \text{ and } y \text{ are congruent spheres)} \end{aligned}$$

$$\begin{aligned} \text{EqD}(x, y, x', y') =_{\text{def}} & \text{SCG}(x, x') \wedge \text{SCG}(y, y') \wedge \neg\text{P}(x, y) \wedge \neg\text{P}(y, x) \wedge \neg\text{P}(x', y') \wedge \neg\text{P}(y', x') \wedge \\ & \exists z, w(\text{ID}(z, x, y) \wedge \text{ID}(w, x', y') \wedge \text{SCG}(z, w)) \end{aligned}$$

( $x, x'$  are congruent spheres and so are  $y, y'$ ;  $x$  and  $y$  are not one part of the other, analogously  $x'$  and  $y'$ ; the centers of  $x, y$  and those of  $x', y'$  are equidistant)

$$\text{MSP}(x, y) =_{\text{def}} \text{S}(x) \wedge \text{P}(x, y) \wedge \forall z((\text{S}(z) \wedge \text{PP}(x, z)) \rightarrow \neg\text{P}(z, y)) \quad (x \text{ is a maximal sphere contained in } y)$$

$$\begin{aligned} \Sigma\text{SS}(x) =_{\text{def}} & \forall y(\text{P}(y, x) \rightarrow \exists s(\text{MSP}(s, x) \wedge \text{O}(s, y)) \wedge \\ & \forall u, w((\text{MSP}(u, x) \wedge \text{MSP}(w, x) \wedge u \neq w) \rightarrow \neg\text{SCG}(u, w))) \\ & \text{(} x \text{ is the sum of a set of pairwise non-congruent spheres)} \end{aligned}$$

$$\begin{aligned} \Sigma\text{CG}(x, y) =_{\text{def}} & \Sigma\text{SS}(x) \wedge \Sigma\text{SS}(y) \wedge \\ & \forall s(\text{MSP}(s, x) \rightarrow \exists s'(\text{MSP}(s', y) \wedge \text{SCG}(s, s'))) \wedge \\ & \forall s(\text{MSP}(s, y) \rightarrow \exists s'(\text{MSP}(s', x) \wedge \text{SCG}(s, s'))) \wedge \\ & \forall s, u, s', u'(\text{MSP}(s, x) \wedge \text{MSP}(u, x) \wedge \text{MSP}(s', y) \wedge \text{MSP}(u', y) \wedge \text{SCG}(s, s') \wedge \text{SCG}(u, u')) \\ & \rightarrow \text{EqD}(s, u, s', u') \\ & \text{(regions } x \text{ and } y \text{ are congruent and they are the sum of two equivalent sets of pairwise non-} \\ & \text{congruent spheres)} \end{aligned}$$

**Lemma 5.3.1** — **T2** is a  $\Phi_{\alpha-\delta}$ -subtheory of **T1**.

*Proof* — We need to prove that  $\llbracket \text{CG}_1(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{CG}(x, y) \rrbracket$ . See Appendix C.1. ♦

### 5.3.2 FROM T2 TO T4

In Appendix C.2, we show that  $\llbracket \text{C}_2^*(x, y) \rrbracket_{\alpha-\delta} = (\text{dist}(X, Y) = 0)$ . From this, we can define parthood by  $\text{P}_2(x, y) =_{\text{def}} \forall z(\text{C}_2^*(z, x) \rightarrow \text{C}_2^*(z, y))$ . The proof that this definition is correct is as the proof of Proposition 5.2.2. Then, Proposition 5.1.1 gives us all the mereological relations used in the following definitions.

$$\text{CCon}_2(c, x, y) =_{\text{def}} \forall a, b(\text{LDist}_2(a, b, x, y) \rightarrow \exists z(\text{SCDiam}_2(z, c) \wedge \text{O}(z, a) \wedge \text{O}(z, b)))$$

where:

$$\begin{aligned} \text{C}_2^*(x, y) =_{\text{def}} & \forall z \exists z'(\text{CG}(z', z) \wedge \text{O}(z', x) \wedge \text{O}(z', y)); \\ \text{SC}_2^*(x) =_{\text{def}} & \forall y, z(\text{SUM}(x, y, z) \rightarrow \text{C}_2^*(y, z)); \end{aligned} \quad (x \text{ is } w\text{-weakly connected})$$

$$\begin{aligned} \text{LEDiam}_2(x, y) =_{\text{def}} & \text{SC}_2^*(y) \wedge \forall a, b((\text{P}(a, x) \wedge \text{P}(b, x)) \rightarrow \exists y'(\text{CG}(y', y) \wedge \text{O}(y', a) \wedge \text{O}(y', b))); \\ & \text{(the diameter of } x \text{ is less than or equal to the diameter of } y \text{ and } y \text{ is } w\text{-weakly connected)} \end{aligned}$$

$$\begin{aligned} \text{SCDiam}_2(x, y) =_{\text{def}} & \text{SC}_2^*(x) \wedge \forall z((\text{P}(y, z) \wedge \text{SC}_2^*(z)) \rightarrow \text{LEDiam}_2(x, z)); \\ & \text{(the diameter of } x \text{ is less than or equal to the diameter of } y \text{ and } x \text{ is } w\text{-weakly connected)} \end{aligned}$$

$$\begin{aligned} \text{LDist}_2(x, y, x', y') =_{\text{def}} & \exists a(\text{SC}_2^*(a) \wedge \text{O}(a, x) \wedge \text{O}(a, y) \wedge \forall a'(\text{CG}(a', a) \rightarrow (\neg\text{O}(a', x') \vee \neg\text{O}(a', y')))). \\ & \text{(the distance of } x \text{ from } y \text{ is strictly smaller than the distance of } x' \text{ from } y') \end{aligned}$$

**Lemma 5.3.2** — **T4** is a  $\Phi_{\alpha-\delta}$ -subtheory of **T2**.

*Proof*— We need to prove that  $\llbracket \text{CCon}_2(c, x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{CCon}(c, x, y) \rrbracket$ . See Appendix C.2.  $\blacklozenge$

### 5.3.3 FROM T4 TO T5

In **T5** the relation **C** is dispensable (see Proposition 5.2.3), therefore it is sufficient to provide an explicit definition of **Closer** in terms of **CCon**. For this we use the definition (DC14) of section 5.1.3:

$$(DC14) \quad \text{Closer}_4(z, x, y) =_{\text{def}} \exists a(\text{CCon}(a, z, x) \wedge \neg \text{CCon}(a, z, y)).$$

**Lemma 5.3.3** — **T5** is a  $\Phi_{\alpha-\delta}$ -subtheory of **T4**.

*Proof*— We need to prove that  $\llbracket \text{Closer}_4(z, x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{Closer}(z, x, y) \rrbracket$ . This follows from Proposition 5.1.7.  $\blacklozenge$

### 5.3.4 FROM T5 TO T3

Here the explicit definitions we need are more complex. The main reason is that we cannot find a way to split the definitions in pieces which correspond to intuitive or already known notions. So, we end up with a relatively long set of conditions that, taken together, provide the correct constraints, although from such a set of conditions one has little hope to recover the intuition about the defined notion.

In the specific case we deal with in this section, we further split the definition of equidistance (**EqD\***) in two cases depending on the dimension of the domain. This is needed for domains of finite regions, like  $\Phi_\gamma$  and  $\Phi_\delta$ . Thus, we provide two definitions of **EqD\***: one for the one-dimensional domains and one for the others.

Since in **T3** the relation **P** is dispensable (Proposition 5.2.2) and in **T5** both **C** and **P** are definable from **Closer** (Proposition 5.2.3 and the definition (DP) of Proposition 5.1.1), the following turns out to be an explicit definition of **Conj** in terms of **Closer**

$$\begin{aligned} \text{Conj}_5(x, y, x', y') =_{\text{def}} & \exists a, b, a', b' (\text{SR}(a) \wedge \text{SR}(b) \wedge \text{SR}(a') \wedge \text{SR}(b') \wedge \\ & \text{C}(a, x) \wedge \text{C}(b, y) \wedge \text{C}(a', x') \wedge \text{C}(b', y') \wedge \text{EqD}^*(a, b, a', b') \wedge \\ & \forall p_a, p_b ((\text{P}(p_a, a) \wedge \text{P}(p_b, b) \wedge \text{EqD}^*(p_a, p_b, a, b)) \rightarrow (\text{C}(p_a, x) \wedge \text{C}(p_b, y))) \wedge \\ & \forall p'_a, p'_b ((\text{P}(p'_a, a') \wedge \text{P}(p'_b, b') \wedge \text{EqD}^*(p'_a, p'_b, a', b')) \rightarrow (\text{C}(p'_a, x') \wedge \text{C}(p'_b, y'))) \end{aligned}$$

where:

**SR**, given by (DSR6) in section 5.1.4, is defined using **C** and **P** only;

**FD<sub>5</sub>** is defined in terms of **Closer** by (DFD5) in section 5.2;

$$\text{Eq}(z, x, y) =_{\text{def}} \neg \text{Closer}(z, x, y) \wedge \neg \text{Closer}(z, y, x); \quad (z \text{ is equidistant from } x \text{ and } y)$$

and **EqD\*** has different definitions in different domains. Then, in  $\Phi_{\alpha-\beta}$  for  $\mathbb{R}^1$  and in  $\Phi_{\alpha-\delta}$  for  $\mathbb{R}^{n>1}$ , we take

$$\begin{aligned} \text{(a) EqD}^*(x, y, x', y') =_{\text{def}} & \text{FD}_5(x) \wedge \text{FD}_5(y) \wedge \text{FD}_5(x') \wedge \text{FD}_5(y') \wedge \\ & \exists z, z' (\text{Eq}(x, y, z) \wedge \text{Eq}(x', y', z') \wedge \text{Eq}(z, x, z') \wedge \text{Eq}(z', x', z)). \quad (x \text{ is as close to } y \text{ as } x' \text{ is to } y') \end{aligned}$$

In  $\Phi_{\gamma-\delta}$  for  $\mathbb{R}^1$  we take:

$$(b) \text{EqD}^*(x, y, x', y') =_{\text{def}} (\text{C}(x, y) \wedge \text{C}(x', y')) \vee \exists z, z' (\text{EC}(z, x) \wedge \text{EC}(z, y) \wedge \text{EC}(z', x') \wedge \text{EC}(z', y') \wedge \text{CG}^*(z, z'))$$

where

$$\text{CG}^s(x, y) =_{\text{def}} \text{FD}_5(x) \wedge \text{FD}_5(y) \wedge \neg \text{C}(x, y) \wedge \exists z_1, z_2, z_3 (\text{EC}(z_2, x) \wedge \text{EC}(z_2, y) \wedge \text{EC}(z_1, x) \wedge \neg \text{C}(z_1, z_2) \wedge \text{EC}(z_3, y) \wedge \neg \text{C}(z_3, z_2) \wedge \text{Eq}(z_2, z_1, z_3));$$

$$\begin{aligned} \text{CG}^*(x, y) =_{\text{def}} & \text{FD}_5(x) \wedge \text{FD}_5(y) \wedge \\ & (x = y \vee \\ & (\text{PO}(x, y) \wedge \exists z_1, z_2 (\text{DIF}(z_1, x, y) \wedge \text{DIF}(z_2, y, x) \wedge \text{CG}^s(z_1, z_2))) \vee \\ & (\text{EC}(x, y) \wedge \neg \exists z ((\text{PP}(z, x) \wedge \text{CG}^s(z, y)) \vee (\text{PP}(z, y) \wedge \text{CG}^s(z, x)))) \vee \\ & (\neg \text{C}(x, y) \wedge \text{CG}^s(x, y))).^{23} \end{aligned}$$

**Lemma 5.3.4** — **T3** is a  $\Phi_{\alpha-\delta}$ -subtheory of **T5**.

*Proof* — We need to prove that  $\llbracket \text{Conj}_5(x, y, x', y') \rrbracket_{\alpha-\delta} = \llbracket \text{Conj}(x, y, x', y') \rrbracket$ . See Appendix C.3. ♦

### 5.3.5 FROM T3 TO T1

Recall from section 5.2. that  $\text{C}_3^*(x, y)$  is defined using  $\text{Conj}$  via (DC3\*), i.e.,

$$(DC3^*) \text{C}_3^*(x, y) =_{\text{def}} \forall z (\text{Conj}(z, z, x, y)).$$

$\text{P}$  (and therefore also  $\text{SUM}$ ) is definable using  $\text{C}_3^*$  (see Proposition 5.2.2) and  $\llbracket \text{C}_3(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{C}(x, y) \rrbracket$  (see Appendix C.4). Therefore we can rely on several definitions introduced in Proposition 5.1.1 (e.g.  $\text{SC}$  and  $\text{TPP}$ ) and on the definition of  $\text{SR}$  introduced in Proposition 5.1.8.

Using these relations we can give an explicit definition of sphere in terms of  $\text{Conj}$ :

$$\begin{aligned} \text{S}_3(x) =_{\text{def}} & \text{FD}_3(x) \wedge \text{SR}(x) \wedge \forall a (\text{LEDiam}_3(a, x) \rightarrow \\ & \exists b (\text{LEDiam}_3(b, a) \wedge \text{P}(b, x) \wedge \forall c, d (\text{EC}(c, x) \wedge \text{EC}(d, x)) \rightarrow (\text{Conj}(b, c, b, d)))) \end{aligned}$$

where

$$\text{SC}_3^*(x) =_{\text{def}} \forall y, z (\text{SUM}(x, y, z) \rightarrow \text{C}_3^*(y, z)); \quad (x \text{ is } w\text{-weakly connected})$$

$$\text{FD}_3(x) =_{\text{def}} \exists x', y, z (\text{SC}_3^*(x') \wedge \text{P}(x, x') \wedge \neg \text{Conj}(x', x', y, z)); \quad (x \text{ has finite diameter})$$

$$\text{C}_3(x, y) =_{\text{def}} \exists z, w (\text{FD}_3(z) \wedge \text{FD}_3(w) \wedge \text{P}(z, x) \wedge \text{P}(w, y) \wedge \text{C}_3^*(z, w))$$

$$\begin{aligned} \text{LEDiam}_3(x, y) =_{\text{def}} & \text{SR}(x) \wedge \text{SR}(y) \wedge \forall a, b ((\text{P}(a, x) \wedge \text{P}(b, x)) \rightarrow \exists a', b' (\text{P}(a', y) \wedge \text{P}(b', y) \wedge \text{Conj}(a, b, a', b'))) \\ & (\text{the diameter of } x \text{ is less than or equal to the diameter of } y \text{ and } x \text{ and } y \text{ are connected}) \end{aligned}$$

**Lemma 5.3.5** — **T1** is a  $\Phi_{\alpha-\delta}$ -subtheory of **T3**.

*Proof* — We need to prove that  $\llbracket \text{S}_3(x) \rrbracket_{\alpha-\delta} = \llbracket \text{S}(x) \rrbracket$ . See Appendix C.4. ♦

<sup>23</sup> In this definition, some conditions are redundant like  $\neg \text{C}(x, y)$ . We include them in the attempt to improve the readability of the formula.

### 5.3.6 FROM T1 TO T6

By Proposition 5.1.2, in **T1**, (DC1) defines **C** for all the structures. Therefore, it is sufficient to provide an explicit definition of **ConvH** in terms of **P** and **S**. We use the additional relation **BTW** which was introduced by Tarski in [56]. As done before, we report only the interpretation. A full description and the related proof of correctness can be found in [56].

$$\llbracket \text{BTW}(x_1, x_2, x_3) \rrbracket_{\alpha-\delta} = \exists c_1, c_2, c_3, r_1, r_2, r_3 (\text{Btw}(c_1, c_2, c_3) \wedge X_1 = \text{ball}(c_1, r_1) \wedge X_2 = \text{ball}(c_2, r_2) \wedge X_3 = \text{ball}(c_3, r_3)).$$

Using the above **BTW**, we can define

$$\text{ConvH}_1(x, y) =_{\text{def}} \text{Conv}(x) \wedge \text{P}(y, x) \wedge \neg \exists z (\text{Conv}(z) \wedge \text{P}(y, z) \wedge \text{PP}(z, x))$$

where

$$\text{Conv}(x) =_{\text{def}} \forall s_1, s_2, s_3 ((\text{P}(s_1, x) \wedge \text{P}(s_2, x) \wedge \text{BTW}(s_3, s_1, s_2)) \rightarrow \text{O}(s_3, x)).$$

**Lemma 5.3.6** — **T6** is a  $\Phi_{\alpha-\delta}$ -subtheory of **T1**.

*Proof* —  $\llbracket \text{ConvH}_1(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{ConvH}(x, y) \rrbracket$ . See Appendix C.5. ♦

### 5.3.7 THE MAIN THEOREM

Now we can state the main result of this paper:

**Main Theorem** —

- (a) **T1–T5** are  $\Phi_{\alpha-\theta}$ -equivalent;
- (b) **T6** is a  $\Phi_{\alpha-\theta}$ -subtheory of **T1–T5**;
- (c) **T1–T4** are conceptually equivalent.

*Proof* — (a) For  $\Phi_{\alpha-\delta}$  the thesis follows from Lemmas 5.3.1 – 5.3.5. For the structures  $\Phi_{\varepsilon-\theta}$ , it follows from our argument in section 4 together with the results of Lemmas 5.3.1 – 5.3.5.

(b) From Lemma 5.3.6 and (a).

(c) From (a) and Definition 2.4. ♦

Note that we do not put constraints on the dimension of the space. Indeed, the result is valid in  $\mathbb{R}^n$  for any positive  $n$ . On the other hand, the result relies on the properties of the considered domains and it might be hard, if possible at all, to extend it to other domains. For example, it is known that **P** cannot be defined from **C** using the definition (DP) in section 5.1 when dealing with atomic theories (see [35, 46]). A similar result holds between **C** and **SR** as given in **T2**.

## 6 FINAL COMMENTS

As we have pointed out in the introduction, a major motivation for this comparison of mereogeometries is the need of evaluating the strength of the mereogeometrical systems in the literature. It is known from the work of Tarski that system **T1** can be used to capture the full system of Euclidian geometry by defining, in second order logic, points to be collections of concentric spheres. This result suggests that theory **T1** is perhaps the strongest system we can look for while remaining within the realm of (region-based) geometry. The most relevant systems in the literature that we have analyzed are formally

equivalent to **T1** in the sense of section 2. We take this fact as evidence that all these theories capture the ‘same’ notion of (mereo)geometry, and that the strength of other systems should be measured with respect to these.

**Definition 6.1** — *A full mereogeometry is a theory which is conceptually equivalent to **T1**.*

Here is an immediate consequence of the Main Theorem

**Corollary 6.1** — ***T1–T4** are full mereogeometries.*

Our comparison does not establish the exact relationship between a full mereogeometry and theory **T6**. It has been argued in [13] that the predicates **C** and **ConvH** do not suffice to obtain what we call here a full mereogeometry. Furthermore, the primitive **ConvH**, at least when interpreted in  $\mathbb{R}^n$ , seems to be naturally related to a (restricted) application of the *Btw* relation (see section 3.1), that is, to a relation which alone is too weak to capture Euclidean geometry (see [57]). These observations make us to believe that **T6** cannot be as strong as **T1**. This is consistent with the results in [19] and matches the conjecture “Mereology + Convexity = Affine Geometry” in [40]. However, we have no direct proof of this and leave the issue as an open question:

**Conjecture** — **T6** is not a full mereogeometry.

We think that the conceptual analysis of mereogeometries presented in this work, even with the limits discussed in section 4, puts order on the relationship among important theories in the literature. In particular, the Main Theorem states that *in the given environment structures* the theories **T1–T5** have the same expressive power. This means that, leaving aside computational issues, there is no real difference among these theories and that, for applicative concerns, the choice of which system to adopt can be safely based on non-logical issues like, for instance, cognitive and modeling adequacy.

We remark here once more why our analysis is not conclusive about the classification of theory **T5** as a full mereogeometry. We have seen that this theory is formally equivalent to **T1–T4** in the frameworks we considered. Nonetheless, our analysis consider also non-formal aspects among which there is the natural domain of interpretation for the theory. Theory **T5** is introduced with a natural domain that we have not considered and we have no proof that this theory is equivalent to the others in an environment with such a domain. Thus, as of now, we cannot claim that **T5** itself is a full mereogeometry. What we can say is that if someone wants to use the formal system **T5** within one of the domains we considered, call this theory **T5'**, then from our result it follows that theory **T5'** is a mereogeometry to all effects.

It is important to note that the definitions we have studied in this paper are all stated in a first order language. Therefore, they can be applied to furnish explicit definitions between (fragments of) the theories. As we have seen, in some cases these definitions are quite complex. The complexity may increase even further if we look for a direct connection between those theories that we did not link explicitly. For example, the definition of the primitives of **T1** in terms of the primitives of **T5** is given indirectly: in the first step we define the primitives of **T3** in **T5** and, then, we use these to define those of **T1**. A complete analysis of these definitions focusing on the complexity of the formulas could highlight important aspects both from the conceptual and the applicative points of view.

Finally, since **T1** is semantically complete with respect to its natural model with domain  $D_\omega$ , our explicit definitions in sections 5.3.1–5.3.5 provide a simple way to obtain a semantically complete axiomatization of all the theories **T2–T5** in the domain  $D_\omega$ . This result is particularly relevant since, as we have seen, many systems are presented in the literature with a partial axiomatization only.

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## APPENDIX A

### BASIC TOPOLOGICAL AND GEOMETRICAL LEMMAS

**Lemma L.1** — *Given a regular set  $X$  in  $\mathbb{R}^n$ , then  $\text{Conx}(X) \rightarrow \text{WConx}(X) \rightarrow \text{WWConx}(X)$  while none of the converse implications hold in general. Furthermore,  $(\text{WWConx}(X) \wedge \text{diam}(X) < +\infty) \rightarrow \text{WConx}(X)$ .*

*Proof* — For  $X$  empty, there is nothing to prove. Otherwise, the first claim holds from the definitions since  $A^\circ \cap B^\circ \neq \emptyset$  implies  $[A] \cap [B] \neq \emptyset$  and the latter implies  $\text{dist}(A, B) = 0$ .

$\text{WConx}(X) \rightarrow \text{Conx}(X)$  fails. For a counterexample in  $\mathbb{R}^2$ , take  $X = A \cup B$  where  $A = \{(x, -y) \mid y > 0, x > 0\}$  and  $B = \{(x, y) \mid y > 0, x > 0\}$ .  $\text{WWConx}(X) \rightarrow \text{WConx}(X)$  fails as well. For a counterexample in  $\mathbb{R}^2$ , take  $X = A \cup B$  where  $A = \{(x, y) \mid y > 1/x > 0\}$  and  $B = \{(x, y) \mid x > 0, y < 0\}$ .

Finally, let  $\text{WWConx}(X)$  with  $\text{diam}(X) < +\infty$ . Fix any pair  $A, B$  satisfying the definition for  $\text{WWConx}(X)$  and consider a sequence  $(a_i)$  in  $A$  and a sequence  $(b_i)$  in  $B$  such that  $\lim_{i \rightarrow \infty} \text{dist}(a_i, b_i) = 0$  (such sequences exist since  $\text{dist}(A, B) = 0$ ). Since  $\text{diam}(X) < +\infty$ , there exist  $a, b \in \mathbb{R}^n$  and subsequences  $(a_j)$  and  $(b_j)$  of  $(a_i)$  and  $(b_i)$ , respectively, such that  $a = \lim_{j \rightarrow \infty} a_j$  and  $b = \lim_{j \rightarrow \infty} b_j$ . Clearly,  $a \in [A]$  and  $b \in [B]$  and, from  $\text{dist}(a, b) = \lim_{j \rightarrow \infty} \text{dist}(a_j, b_j) = 0$ , we conclude  $a = b$ , that is,  $[A] \cap [B] \neq \emptyset$ . ♦

**Lemmas L.2/L.6** — *Let  $X$  and  $Y$  be arbitrary sets in a topological space, then:*

**L.2**  $X^\circ \subseteq X \subseteq [X]$

**L.3**  $[X \cup Y] = [X] \cup [Y]$  and  $[X \cap Y] \subseteq [X] \cap [Y]$

**L.4**  $(X \cap Y)^\circ = X^\circ \cap Y^\circ$  and  $X^\circ \cup Y^\circ \subseteq (X \cup Y)^\circ$

**L.5** If  $X \subseteq Y$ , then  $X^\circ \subseteq Y^\circ$  and  $[X] \subseteq [Y]$

**L.6** If  $[X] \cap [Y] = \emptyset$ , then  $(X \cup Y)^\circ = X^\circ \cup Y^\circ$

*Proof* — Lemmas L.2 – L.5 are basic topological results. (see for instance [38]).

Regarding L.6: from L.4 we know  $X^\circ \cup Y^\circ \subseteq (X \cup Y)^\circ$ , we need to show  $X^\circ \cup Y^\circ \supseteq (X \cup Y)^\circ$ . If  $X$  or  $Y$  is empty, there is nothing to prove. Assume they are both non-empty and  $x \in (X \cup Y)^\circ$ , then  $x \in X \cup Y$ . Suppose  $x \in X$ . If  $x \in X^\circ$ , then  $x \in X^\circ \cup Y^\circ$  and we are done. If  $x \notin X^\circ$ , i.e.  $x \in \partial(X)$ , then  $x \notin [Y]$  since  $[X] \cap [Y] = \emptyset$ . Thus, there exists a neighborhood of  $x$ , say  $I(x)$ , such that  $I(x) \cap Y = \emptyset$ . From  $x \in \partial(X)$  and  $x \notin [Y]$ ,  $I(x) \cap (X \cup Y) \neq \emptyset$  and  $I(x) \cap \sim(X \cup Y) \neq \emptyset$ . This happens for any neighborhood of  $x$  contained in  $I(x)$ , thus  $x \in \partial(X \cup Y)$  contradicting the hypothesis  $x \in (X \cup Y)^\circ$ . Finally, we have  $x \notin \partial(X)$ . Thus,  $x \in X$  implies  $x \in X^\circ$ . One can prove analogously that  $x \in Y$  implies  $x \in Y^\circ$ . From these results,  $x \in (X \cup Y)^\circ$  implies  $x \in X^\circ \cup Y^\circ$  and we are done. ♦

**Lemma L.7** — *Let  $X$  and  $Y$  be open regular sets in a topological space  $T$ , then  $X \cap Y$ ,  $[X \cup Y]^\circ$ , and  $\sim[X]$  are open regular sets. Let  $X$  and  $Y$  be closed regular sets in a topological space  $T$ , then  $[(X \cap Y)^\circ]$ ,  $X \cup Y$ , and  $\sim(X^\circ)$  are closed regular sets.*

*Proof* — Directly from the fact that regular open sets form a Boolean algebra with  $1 = T$ ,  $0 = \emptyset$ ,  $X \cdot Y = X \cap Y$ ,  $X + Y = [X \cup Y]^\circ$ , and  $\sim X = \sim[X]$  (see [6]). Analogously for the regular closed sets. ♦

**Lemma L.8** — *Given two non-empty regular sets  $X$  and  $Y$  in  $\mathbb{R}^n$ :*

$$\text{dist}(X^\circ, Y^\circ) = \text{dist}(X, Y) = \text{dist}([X], [Y])$$

*Proof* — From L.2 and from the definition of distance we have  $dist([X], [Y]) \leq dist(X, Y) \leq dist(X^\circ, Y^\circ)$ . For the other direction, suppose (by contradiction) that  $dist([X], [Y]) < dist(X^\circ, Y^\circ)$ , i.e.  $i = \inf\{dist(x, y) \mid x \in [X] \wedge y \in [Y]\} < \inf\{dist(x, y) \mid x \in X^\circ \wedge y \in Y^\circ\} = i'$ . Then, there exists  $d > 0$  such that  $i' = i + d$ . From the definition of distance between sets, for all  $\varepsilon > 0$  there exist  $x \in [X]$  and  $y \in [Y]$  such that  $dist(x, y) < i + \varepsilon$ , fix  $\varepsilon = d/3$ , then  $dist(x, y) < i'$ . Thus,  $ball(x, \varepsilon) \cap X^\circ = \emptyset$  or  $ball(y, \varepsilon) \cap Y^\circ = \emptyset$  (otherwise we could find points  $x' \in X^\circ$  and  $y' \in Y^\circ$  with  $dist(x', y') < i'$ ). From this result,  $x \notin [X]$  or  $y \notin [Y]$ . Contradiction. Then,  $dist([X], [Y]) \geq dist(X^\circ, Y^\circ)$ , from which the thesis follows.  $\blacklozenge$

**Lemma L.9** — *Given a non-empty regular set  $X$  in  $\mathbb{R}^n$ :*

$$diam(X^\circ) = diam(X) = diam([X])$$

*Proof* — From L.2 and from the definition of diameter,  $diam(X^\circ) \leq diam(X) \leq diam([X])$ . For the other direction, suppose (by contradiction) that  $diam([X]) > diam(X^\circ)$ , i.e.  $s = \sup\{dist(x, y) \mid x, y \in [X]\} > \sup\{dist(x, y) \mid x, y \in X^\circ\} = s'$ . Then, there exists  $d > 0$ , such that  $s - d = s'$ . Since for all  $\varepsilon > 0$  there exist  $x, y \in [X]$  such that  $dist(x, y) > s - \varepsilon$ , fix  $\varepsilon = d/3$ , then  $dist(x, y) > s'$  and so  $ball(x, \varepsilon) \cap X^\circ = \emptyset$  or  $ball(y, \varepsilon) \cap X^\circ = \emptyset$ . From this result,  $x \notin [X]$  or  $y \notin [X]$ . Contradiction. Thus,  $diam([X]) \leq diam(X^\circ)$  from which the thesis follows.  $\blacklozenge$

**Lemma L.10** — *Given two non-empty regular sets  $X$  and  $Y$  in  $\mathbb{R}^n$ :*

$$diam(X \cup Y) \leq dist(X, Y) + diam(X) + diam(Y)$$

*Proof* — This result follows easily from the triangular inequality.  $\blacklozenge$

**Lemma L.11** — *Given two open regular sets  $X$  and  $Y$  in  $\mathbb{R}^n$ . If  $X$  and  $Y$  have finite diameter, then  $X \cap Y$ , and  $[X \cup Y]^\circ$  are open regular sets with finite diameters.*

*Proof* — From L.7,  $X \cap Y$  and  $[X \cup Y]^\circ$  are open regular sets. In the first case, the condition on the diameter follows from  $(X \cap Y) \subseteq X$ , while, in the second case, from L.8, L.9, and L.10.  $\blacklozenge$

**Lemma L.12** — *Given two non-empty regular sets  $X, Y \subseteq \mathbb{R}^n$  with at least one of finite diameter:*

$$\exists x, y (x \in \partial(X) \wedge y \in \partial(Y) \wedge dist(x, y) = dist(X, Y))$$

*Proof* — First we show  $\exists x, y (x \in [X] \wedge y \in [Y] \wedge dist(x, y) = dist(X, Y))$ .

Assume both  $X$  and  $Y$  have finite diameter. By definition of  $dist(X, Y)$ , one can find a sequence  $(a_i)$  in  $X$  and a sequence  $(b_i)$  in  $Y$  such that  $\lim_{i \rightarrow \infty} dist(a_i, b_i) = dist(X, Y)$ . Since  $diam(X), diam(Y) < +\infty$ , there exist  $a, b \in \mathbb{R}^n$  and subsequences  $(a_j)$  and  $(b_j)$  of  $(a_i)$  and  $(b_i)$ , respectively, such that  $a = \lim_{j \rightarrow \infty} a_j$  and  $b = \lim_{j \rightarrow \infty} b_j$ . Clearly,  $a \in [X]$ ,  $b \in [Y]$ , and  $dist(a, b) = \lim_{j \rightarrow \infty} dist(a_j, b_j) = dist(X, Y)$ .

If  $diam(Y) = +\infty$ , we proceed as before to isolate  $a \in [X]$ , since  $[X]$  is compact. Then, we consider sequence  $(b_i)$  in  $Y$  such that  $\lim_{i \rightarrow \infty} dist(a, b_i) = dist(X, Y)$ . For any positive  $r$ ,  $(b_i) \cap \{y \in [Y] \mid dist(y, a) < dist(X, Y) + r\}$  contains an infinite subsequence of  $(b_i)$ , call it  $(c_i)$ . Then, there exists  $c = \lim_{j \rightarrow \infty} c_j$  for some subsequence  $(c_j)$  of  $(c_i)$ . Clearly,  $c \in [Y]$  and  $dist(a, c) = dist(X, Y)$ .

Now assume we isolated  $x$  and  $y$  satisfying  $\exists x, y (x \in [X] \wedge y \in [Y] \wedge dist(x, y) = dist(X, Y))$ . It is easy to see that if  $x \in X^\circ$ , in  $\mathbb{R}^n$  one can find  $x' \in [X]$  such that  $dist(x', y) < dist(x, y)$ . Thus  $x \in \partial(X)$ . Similarly for  $y$ .  $\blacklozenge$

**Lemma L.13** — *Given a non-empty regular set  $X$  in  $\mathbb{R}^n$ :*

$$WConx(X) \rightarrow \forall d(0 \leq d < diam(X) \rightarrow \exists x,y(x,y \in X \wedge dist(x,y) = d))$$

If  $X$  has finite diameter:

$$\exists x,y(x,y \in [X] \wedge dist(x,y) = diam(X))$$

*Proof* — The result is trivial for  $d = 0$ . For  $d > 0$ , let  $X$  be open (for  $X$  closed it follows from this). Fix  $d \in (0, diam(X))$  and fix  $x, z \in X$  such that  $dist(x, z) = d'$  with  $d < d' < diam(X)$  (these points exist from the definition of  $diam$ ). Let  $A = \{y \in X \mid dist(x, y) \leq d\}$  and let  $B = \{y \in X \mid dist(x, y) > d\}$ . Clearly,  $X = A \cup B$  and both  $A$  and  $B$  are non-empty since  $x \in A$  and  $z \in B$ . If there exists  $u \in A$  such that  $dist(x, u) = d$ , we are done. Otherwise we have that  $A$  is open in  $\mathbb{R}^n$  since  $A$  is open in  $X$  and  $X$  is open in  $\mathbb{R}^n$ . Also, note that  $B$  is open, since, from the definition,  $B$  is open in  $X$ . From  $WConx(X)$ , there exist  $v \in [A] \cap [B]$ . Thus  $dist(x, v) = d$ .

Let  $O_x = ball(x, \delta)$ , for some  $\delta$ , such that  $[O_x] \subset X$  and fix  $y \in B$  such that  $dist(y, v) < \delta/2$  (this point exists since  $v \in \partial(B)$ ). Fix the line through  $y$  and  $x$ , call it  $L$ . Let  $\{x_1, x_2\} = \partial(L \cap [O_x])$  with  $d_1 = dist(y, x_1) < d_2 = dist(y, x_2)$ . Then  $d_1 < d < d_2$ . Since  $[y, x_2] \subset L$  is compact, the function  $dist$  on  $[y, x_2] \times \{y\}$  assumes all values in  $[0, d_2]$ . Since  $dist$ , on  $[y, x_1] \times \{y\}$ , assumes all values in  $[0, d_1]$  and it is strictly increasing, there exists  $x^* \in [x_1, x_2] \subset (L \cap [O_x])$  such that  $dist(x^*, y) = d$ . Since  $x^*, y \in X$  we are done.

For the second claim. By definition of  $diam(X)$ , one can find two sequences  $(a_i)$  and  $(b_i)$  in  $X$  such that  $\lim_{i \rightarrow \infty} dist(a_i, b_i) = diam(X)$ . Since  $diam(X) < +\infty$ , there exist  $a, b \in \mathbb{R}^n$  and subsequences  $(a_j)$  and  $(b_j)$  of  $(a_i)$  and  $(b_i)$ , respectively, such that  $a = \lim_{j \rightarrow \infty} a_j$  and  $b = \lim_{j \rightarrow \infty} b_j$ . Clearly,  $a \in [X]$ ,  $b \in [X]$  and  $dist(a, b) = \lim_{j \rightarrow \infty} dist(a_j, b_j) = diam(X)$ . ♦

**Lemma L.14** — Given two non-empty regular sets  $X, Y \subseteq \mathbb{R}^n$  with finite diameter:

$$(WConx(X) \wedge WConx(Y) \wedge dist(X, Y) = 0) \rightarrow WConx(X \cup Y)$$

*Proof* — If not, then there exist  $A^\circ$  and  $B^\circ$  non-empty such that  $(X \cup Y)^\circ = A^\circ \cup B^\circ$  and  $[A] \cap [B] = \emptyset$ . If  $A^\circ \cap X^\circ \neq \emptyset$  and  $B^\circ \cap X^\circ \neq \emptyset$ , then  $A^\circ \cap X^\circ$  and  $B^\circ \cap X^\circ$  contradict  $WConx(X)$ . Thus, either  $X^\circ \subseteq A^\circ$  or  $X^\circ \subseteq B^\circ$ . Similarly for  $Y$ . Thus we have  $X^\circ = A^\circ$  or  $X^\circ = B^\circ$ , and  $Y^\circ = A^\circ$  or  $Y^\circ = B^\circ$ .

Among the two cases (1)  $X^\circ = Y^\circ = A^\circ$  (or  $X^\circ = Y^\circ = B^\circ$ ) and (2)  $X^\circ = A^\circ$  and  $Y^\circ = B^\circ$  (or vice versa), the first contradicts the assumption that both  $A^\circ$  and  $B^\circ$  are non-empty. For case (2), assume  $X^\circ = A^\circ$  and  $Y^\circ = B^\circ$  (the other option is similar). From  $dist(X, Y) = 0$  and L.8, we have  $dist(A, B) = 0$ . From L.1, for any regular set  $Z$ , if  $WConx(Z)$ , then  $WWConx(Z)$ . In particular,  $WWConx(X)$  and  $WWConx(Y)$ . Our argument above, applies to all possible pairs of non-empty of sets  $A^\circ$  and  $B^\circ$  such that  $(X \cup Y)^\circ = A^\circ \cup B^\circ$ . This proves that  $WWConx(X \cup Y)$  holds. Since L.10 implies  $diam(X \cup Y)$  is finite, it suffices to recall L.1 to conclude  $WConx(X \cup Y)$ . ♦

**Lemma L.15** — Given two non-empty regular sets  $X, Y \subseteq \mathbb{R}^n$  with finite diameter:<sup>24</sup>

$$[X] \cap [Y] \neq \emptyset \Leftrightarrow dist(X, Y) = 0$$

*Proof* — From L.12 and the definition of  $dist$ . ♦

**Lemma L.16** — Given an open regular set  $X$  in  $\mathbb{R}^n$ ,  $Conv(X) \rightarrow Conx(X)$ .

<sup>24</sup> Note that this result provides an indirect proof of  $(WWConx(X) \wedge diam(X) < +\infty) \rightarrow WConx(X)$ .

*Proof* — If  $X$  is empty, there is nothing to prove. If  $X$  is non-empty, we proceed by contradiction. Let  $X^\circ = A^\circ \cup B^\circ$  with  $A^\circ, B^\circ$  both non-empty and  $A \cap B = \emptyset$ . Since  $X$  is convex, for any  $a \in A$  and  $b \in B$  the segment  $(a, b)$  should be contained in  $A \cup B$ . But  $(a, b) \cap A$  and  $(a, b) \cap B$  are open non-overlapping segments and an open segment cannot be split in two open non-overlapping sub-segments. This implies that  $(a, b) \not\subseteq X$  and contradicts the assumption  $\text{Conv}(X)$ .  $\blacklozenge$

**Lemma L.17** — *Given a regular connected (Conx) region  $X$  in  $\mathbb{R}^n$  and a point  $p \in [X]$ , there exists  $Y \subseteq X$  such that  $\text{Conx}(Y)$ ,  $\text{diam}(Y) < +\infty$ , and  $p \in [Y]$ .*

*Proof* — If  $X$  is with finite diameter, there is nothing to show. Let  $\text{diam}(X) = +\infty$  and  $X$  open (if  $X$  is closed consider  $X^\circ$ ). If  $p \in X$ , there exists  $C_r = \text{ball}(p, r) \subseteq X$  for some  $r$  and we are done. If  $p \in \partial(X)$ , let  $C_r = \text{ball}(p, r)$  and consider  $Z_r = C_r \cap X$ . We show that, for some  $r$ ,  $Z_r$  is connected. By contradiction, let's assume that, for all  $r$ ,  $Z_r$  is not connected. Since  $Z_1$  is open, given a point  $q \in Z_1$ , it is possible to find a (connected) neighborhood  $Q$  of  $q$  contained in  $Z_1$ . Fix  $q$  and let  $A_1$  be the maximal connected part of  $Z_1$  that contains  $q$ . Build two sequences of regions  $A_1, A_2, \dots$  and  $B_1, B_2, \dots$ , such that  $A_n$  is the maximal connected open region part of  $Z_n = \text{ball}(p, n)$ ,  $A_n \subseteq A_{n+1}$ , and  $B_n = Z_n - A_n$ . Since  $Z_n$  is not connected by hypothesis and  $A_n$  is a maximal connected region, we have that  $A_n$  and  $B_n$  are open,  $Z_n = A_n \cup B_n$ , and  $A_n \cap B_n = \emptyset$ . Since for  $n \rightarrow +\infty$ ,  $Z_n = X$ , we have that  $X = (\cup A_n) \cup (\cup B_n)$ . Since, for each  $n$ ,  $A_n$  and  $B_n$  are open and  $A_n \cap B_n = \emptyset$  we get  $(\cup A_n) \cap (\cup B_n) = \emptyset$ , i.e.  $X$  is not connected. Contradiction.  $\blacklozenge$

**Lemma L.18** — *Given a regular set  $X$  in  $\mathbb{R}^n$ :*

$$WW\text{Conx}(X) \rightarrow \forall d(0 \leq d < \text{diam}(X) \rightarrow \exists x, y(x, y \in X \wedge \text{dist}(x, y) = d))$$

*Proof* — If  $WW\text{Conx}(X)$ , then this claim reduces to L.13. If  $X$  is finite, then it follows from L.1 and L.13. Now, assume  $X$  is infinite,  $WW\text{Conx}(X)$  and not  $W\text{Conx}(X)$ . We prove something even stronger, that is, we show that each regular set  $X$  with infinite diameter contains a regular set  $Y \subseteq X$  whose diameter is also infinite and such that  $W\text{Conx}(Y)$ . From this, applying L.13 to  $Y$ , we conclude.

Fix a rational<sup>25</sup>  $y_0 \in X^\circ$  and let  $Z_0 = \cup C$  where  $C \subseteq X$ ,  $W\text{Conx}(C)$  and  $y_0 \in C^\circ$ . First, since  $y_0$  is internal, there exists  $D \subseteq X$  open ball centered in  $y_0$  and, from L.16,  $W\text{Conx}(D)$ , thus  $Z_0 \neq \emptyset$ . We show that  $W\text{Conx}(Z_0)$ . Suppose  $W\text{Conx}(Z_0)$  fails, then there exist  $A^\circ$  and  $B^\circ$  non-empty such that  $Z_0^\circ = A^\circ \cup B^\circ$  and  $[A] \cap [B] = \emptyset$ . Since  $W\text{Conx}(C)$  and  $C^\circ \subseteq Z_0^\circ = A^\circ \cup B^\circ$ , for all  $C$  in  $\cup C$ , we have either  $A^\circ \cap C^\circ = \emptyset$  or  $B^\circ \cap C^\circ = \emptyset$ . Then,  $C^\circ \subseteq A^\circ$  or  $C^\circ \subseteq B^\circ$ . However,  $y_0 \in C$  for all such  $C$ , thus either for all  $C$ ,  $C^\circ \subseteq A^\circ$  or, for all  $C$ ,  $C^\circ \subseteq B^\circ$ . That is, either  $A^\circ = \emptyset$  or  $B^\circ = \emptyset$ . Contradiction. This proves  $W\text{Conx}(Z_0)$ . Note that  $Z_0$  is closed in  $X$ . Indeed, if  $x \in X \cap \partial(Z_0)$ , then  $W\text{Conx}(Z_0^\circ \cup \{x\})$  holds since the only hope to find a counterexample is by splitting  $(Z_0^\circ \cup \{x\})^\circ$  into  $Z_0^\circ$  and  $\{x\}^\circ$ , but the latter set is empty. Thus  $Z_0^\circ \cup \{x\}$  itself is one of the  $C$ 's considered in the construction of  $Z_0$ . This implies  $x \in Z_0$  and so  $Z_0$  is closed in  $X$ .

If  $\text{diam}(Z_0) = +\infty$ , we are done. Otherwise, consider  $X - Z_0$  and let  $y_1$  be a rational in  $(X - Z_0)^\circ$ . We repeat the construction above to find  $Z_1$  maximal in  $X - Z_0$  such that  $W\text{Conx}(Z_1)$ . If  $\text{diam}(Z_1) = +\infty$ , we are done. Otherwise, consider  $X - (Z_0 \cup Z_1)$  and let  $y_2$  be a rational point in set  $(X - (Z_0 \cup Z_1))^\circ$ . Again, we repeat the construction above to find  $Z_2$  maximal in  $X - (Z_0 \cup Z_1)$  such that  $W\text{Conx}(Z_2)$ . We proceed in this way till we find a set  $Z_i$  with  $W\text{Conx}(Z_i)$  and  $\text{diam}(Z_i) = +\infty$ , or till we cover all of  $X^\circ$ . In the first case, we are done because of L.13. We show that the latter case cannot happen. Indeed, in the latter case we obtain a sequence of countably many regular sets  $Z_i$  (covering  $X^\circ$  since the rationals are dense in  $\mathbb{R}^n$ ) such that, for

<sup>25</sup> A point in  $\mathbb{R}^n$  is rational if all its coordinates are rational numbers.

all  $i$ ,  $WConx(Z_i)$  and  $diam(Z_i) < +\infty$ . Fix  $Z_0$ . If  $dist(Z_0, Z_i) \neq 0$  for all  $Z_i \neq Z_0$ , it suffices to put  $Z_0 = A$  and  $\cup_{i>0} Z_i = B$  to find a contradiction to  $WWConx(X)$ . Let  $Z_{i_1}$  be such that  $dist(Z_0, Z_{i_1}) = 0$  and let  $Z_0 \cup Z_{i_1} = U_1$ . By L.14,  $WConx(U_1)$ . As before, there exists  $Z_{i_2}$  such that  $dist(U_1, Z_{i_2}) = 0$ . Let  $U_1 \cup Z_{i_2} = U_2$ . By L.14,  $WConx(U_2)$ . We proceed in this way constructing sets  $U_r = U_{r-1} \cup Z_{i_r}$ . Let  $U_\infty = \cup U_i$ . First, note that if there exists  $i$  such that  $Z_i \not\subseteq U_\infty$ , then we can reapply the argument above to get a contradiction to  $WWConx(X)$ . Thus,  $U_\infty = \cup Z_i$ . By construction, we have  $WConx(U_\infty)$ . However,  $U_\infty^\circ = \cup Z_i^\circ = X^\circ$  and since each  $Z_i$  is closed in  $X$ , we actually have  $U_\infty = \cup Z_i = X$ . This contradicts the assumption that  $WConx(X)$  fails. Thus, either that assumption is wrong (i.e.  $WConx(X)$  holds) or for some  $i$ ,  $WConx(Z_i)$  and  $diam(Z_i) = +\infty$ , which is what we needed to prove.  $\blacklozenge$

**Lemma L.19** — *Given a set  $X$  in  $\mathbb{R}^n$ , if  $Y$  is the smallest convex set containing  $X$  (the convex-hull of  $X$ ), then  $diam(X) = diam(Y)$ .*

*Proof*—If  $X$  is empty, there is nothing to prove. If  $X$  is non-empty, we proceed by contradiction. Since  $X \subseteq Y$  we need to consider only case:  $diam(X) < diam(Y)$ . From this assumption, there exist  $x \in Y - X$  and  $y \in Y$  such that  $dist(x, y) > diam(X)$ . By the definition of  $Y$ , since  $x \in Y - X$ , there exist  $a, b \in X$  such that  $x$  is between  $a$  and  $b$ . Consider the ball  $B$  of radius  $r = dist(x, y)$  centered at  $y$ , and the line  $l$  through  $a, x$ , and  $b$ . Since  $x \in l$ , there are only two cases to consider:  $l$  is tangent to  $B$  or  $l$  intersects  $B$ . In the first case,  $dist(a, y) > r = dist(x, y)$ , contradicting the assumption. In the latter case, by the so-called Pasch axiom of Euclidean geometry, at least one between  $a$  and  $b$  has to lie outside  $C$ . Thus, we reached a contradiction again.  $\blacklozenge$

## APPENDIX B

### B.1 PROOF OF PROPOSITION 5.1.1

**(DP)**  $P^*(x, y) =_{\text{def}} \forall w(C(x, w) \rightarrow C(y, w))$  and  $[[P^*(x, y)]]_{\alpha-\delta} = X \subseteq Y$ .

*Proof*— We have to prove that in all the domains:

$$X \subseteq Y \iff \forall W([W] \cap [X] \neq \emptyset \rightarrow [W] \cap [Y] \neq \emptyset)$$

( $\Rightarrow$ ) Trivial.

( $\Leftarrow$ ) (By contradiction) Assume  $\neg(X \subseteq Y)$ . Since  $X$  and  $Y$  are both open, then  $\neg(X \subseteq [Y])$  by L.2 and L.5. It suffices to fix an open ball  $W$  contained in  $X - [Y]$ . Such a  $W$  exists because  $X - [Y]$  is open and non-empty.  $\blacklozenge$

**(DPP)**  $PP(x, y) =_{\text{def}} P(x, y) \wedge \neg P(y, x)$  and  $[[PP(x, y)]]_{\alpha-\delta} = X \subset Y$ .

*Proof*— This follows from the obvious equivalence:  $(X \subseteq Y \wedge \neg(Y \subseteq X)) \iff X \subset Y$ .  $\blacklozenge$

**(DO)**  $O(x, y) =_{\text{def}} \exists z(P(z, x) \wedge P(z, y))$  and  $[[O(x, y)]]_{\alpha-\delta} = X \cap Y \neq \emptyset$ .

*Proof*— We have to prove that in all the domains:

$$\exists Z(Z \subseteq X \wedge Z \subseteq Y) \iff X \cap Y \neq \emptyset$$

( $\Rightarrow$ ) From the hypothesis  $Z \subseteq (X \cap Y)$ . Since  $Z$  is non-empty in all domains, one has  $X \cap Y \neq \emptyset$ .

( $\Leftarrow$ ) It suffices to fix  $Z$  open ball contained in  $A = X \cap Y$ . (Such a ball exists because, by L.7,  $A$  is an open set of  $\mathbb{R}^n$ .)  $\blacklozenge$

**(DPO)**  $PO(x, y) =_{\text{def}} O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$  and

$$\llbracket \text{PO}(x, y) \rrbracket_{\alpha-\delta} = X \cap Y \neq \emptyset \wedge \neg(X \subseteq Y) \wedge \neg(Y \subseteq X).$$

*Proof*— Trivial. ♦

$$\mathbf{(D+)} \quad \text{SUM}(z, x, y) =_{\text{def}} \forall w(\mathbf{O}(w, z) \leftrightarrow (\mathbf{O}(w, x) \vee \mathbf{O}(w, y))) \text{ and } \llbracket \text{SUM}(z, x, y) \rrbracket_{\alpha-\delta} = (Z = [X \cup Y]^\circ);$$

*Proof*— We have to prove that in all the domains:

$$\forall W(W \cap Z \neq \emptyset \leftrightarrow (W \cap X \neq \emptyset \vee W \cap Y \neq \emptyset)) \iff Z = [X \cup Y]^\circ$$

( $\Rightarrow$ ) Since for all non-empty  $W \subseteq X$  we have  $W \cap X \neq \emptyset$ , then  $W \cap Z \neq \emptyset$ . Thus,  $X \subseteq Z$ . Analogously, we conclude  $Y \subseteq Z$ . Thus,  $X \cup Y \subseteq Z$ . But  $Z$  is a regular open region, then  $[X \cup Y]^\circ \subseteq Z$ . If  $W \cap Z = \emptyset$ , then  $W \cap X = \emptyset$  and  $W \cap Y = \emptyset$ . Thus,  $Z$  is contained in the smallest regular open region that contains both  $X$  and  $Y$ , i.e.,  $[X \cup Y]^\circ$ .

( $\Leftarrow$ ) Substitute  $[X \cup Y]^\circ$  for  $Z$  in the left-hand side. Then

( $\rightarrow$ ) From L.2,  $W \cap [X \cup Y]^\circ \neq \emptyset$  implies that  $W \cap [X \cup Y] \neq \emptyset$ . From L.3,  $[X \cup Y] = [X] \cup [Y]$  and then  $W \cap [X] \neq \emptyset$  or  $W \cap [Y] \neq \emptyset$ . Since  $W, X, Y$  are open and regular, we have  $W \cap X \neq \emptyset$  or  $W \cap Y \neq \emptyset$ .

( $\leftarrow$ )  $X \cup Y$  is open and then  $(X \cup Y) \subseteq [X \cup Y]^\circ$  (from the definition of the interior operator). Since  $(W \cap X \neq \emptyset \vee W \cap Y \neq \emptyset) \leftrightarrow W \cap (X \cup Y) \neq \emptyset$  and  $(X \cup Y) \subseteq [X \cup Y]^\circ$ , then  $W \cap [X \cup Y]^\circ \neq \emptyset$ . ♦

$$\mathbf{(D-)} \quad \text{DIF}(z, x, y) =_{\text{def}} \forall w(\mathbf{P}(w, z) \leftrightarrow (\mathbf{P}(w, x) \wedge \neg \mathbf{O}(w, y))) \text{ and } \llbracket \text{DIF}(z, x, y) \rrbracket_{\alpha-\delta} = (Z = X - [Y]);$$

*Proof*— We have to prove that in all the domains:

$$\forall W(W \subseteq Z \leftrightarrow (W \subseteq X \wedge W \cap Y = \emptyset)) \iff Z = X - [Y]$$

( $\Rightarrow$ ) Let  $W = X - [Y]$ . Then,  $W \subseteq X$  and  $W \cap Y = \emptyset$ . Thus,  $X - [Y] \subseteq Z$ . For the other inclusion, let  $W \cap (X - [Y]) = \emptyset$ . Then,  $W \subseteq X$  or  $W \cap Y = \emptyset$  fails. Thus,  $W \subseteq Z$  fails as well. Since this happens for all  $W$  satisfying this property, we conclude  $Z \subseteq X - [Y]$ .

( $\Leftarrow$ ) Substitute  $X - [Y]$  for  $Z$  in the left-hand side. Since  $W, X, Y$  are all open and regular,  $W \cap Y = \emptyset$  iff  $W \cap [Y] = \emptyset$ . The equivalence follows. ♦

$$\mathbf{(DIP)} \quad \text{IP}(x, y) =_{\text{def}} \mathbf{P}(x, y) \wedge \forall z(\mathbf{C}(z, x) \rightarrow \mathbf{O}(z, y)) \text{ and } \llbracket \text{IP}(x, y) \rrbracket_{\alpha-\delta} = [X] \subseteq Y.$$

*Proof*— We have to prove that in all the domains:

$$(X \subseteq Y \wedge \forall Z([Z] \cap [X] \neq \emptyset \rightarrow Z \cap Y \neq \emptyset)) \iff [X] \subseteq Y$$

( $\Rightarrow$ ) (By contradiction) Suppose  $\neg([X] \subseteq Y)$ . If  $\neg(X \subseteq Y)$  we are done. Otherwise, note that  $Y \neq \mathbf{R}^n$ . From L.7,  $A = \neg[Y]$  is a regular non-empty open set. From  $\neg([X] \subseteq Y)$  and  $X \subseteq Y$ , there exists a point  $x \in \partial(X)$  and  $x \notin Y$ . Since  $x \notin Y$ ,  $x \in [A]$ . Fix a positive number  $r$  and let  $Z = \text{Ball}(x, r) \cap A$ . Then,  $Z \cap Y = \emptyset$ . From L.7,  $Z$  is regular open. Also,  $Z$  is non-empty since  $x \in \partial(X)$ ,  $x \notin Y$  and  $Y$  is regular. From L.4,  $x \in [Z]$ . Then,  $Z \cap Y = \emptyset$  and  $[Z] \cap [X] \neq \emptyset$ . Contradiction.

( $\Leftarrow$ ) From L.2,  $[X] \subseteq Y$  implies  $X \subseteq Y$ . Now suppose  $[Z] \cap [X] \neq \emptyset$  and  $[X] \subseteq Y$ , then we have  $[Z] \cap Y \neq \emptyset$ , i.e.,  $\exists p(p \in [Z] \wedge p \in Y)$ . If  $p \in Z$ , then  $Z \cap Y \neq \emptyset$ . If  $p \in \partial(Z)$ , then there exists a neighborhood of  $p$ , say  $I(p)$ , such that  $I(p) \subseteq Y$  and  $I(p) \cap Z \neq \emptyset$ . This means that there exists a point  $p' \in I(p) \cap Z$  and so  $Z \cap Y \neq \emptyset$ . ♦

$$\mathbf{(DTPP)} \quad \text{TPP}(x, y) =_{\text{def}} \mathbf{PP}(x, y) \wedge \exists z(\mathbf{EC}(z, x) \wedge \mathbf{EC}(z, y)) \text{ and } \llbracket \text{TPP}(x, y) \rrbracket_{\alpha-\delta} = X \subset Y \wedge \partial(X) \cap \partial(Y) \neq \emptyset;$$

*Proof*— We have to prove that in all the domains:

$$X \subset Y \wedge \partial(X) \cap \partial(Y) \neq \emptyset \iff X \subset Y \wedge \exists Z([Z] \cap [X] \neq \emptyset \wedge Z \cap X = \emptyset \wedge [Z] \cap [Y] \neq \emptyset \wedge Z \cap Y = \emptyset)$$

( $\Rightarrow$ ) Let  $p \in \{\partial(X) \cap \partial(Y)\}$ . The thesis follows considering an open ball  $Z$  such that  $p \in [Z]$ , and  $Z \cap Y = \emptyset$ .

( $\Leftarrow$ ) From the hypothesis we have that  $\partial(Z) \cap \partial(X) \neq \emptyset$  and  $\partial(Z) \cap \partial(Y) \neq \emptyset$ . From L.5 and  $X \subset Y$  it follows that  $[X] \subset [Y]$ , therefore  $\partial(X) \cap \partial(Y) \neq \emptyset$ .  $\blacklozenge$

**(DSC)**  $\text{SC}(x) =_{\text{def}} \forall y, z (\text{SUM}(x, y, z) \rightarrow \text{C}(y, z))$  and  $\llbracket \text{SC}(x) \rrbracket_{\alpha-\delta} = \text{WConx}(X)$ .

*Proof*— From the definition of  $\text{WConx}$  and considering the given domains,  $\text{WConx}(X)$  stands for  $\forall Y, Z (X = Y \cup Z \rightarrow [Y] \cap [Z] \neq \emptyset)$ , thus we prove that:

$$\forall Y, Z (X = [Y \cup Z]^\circ \rightarrow [Y] \cap [Z] \neq \emptyset) \iff \forall Y', Z' (X = Y' \cup Z' \rightarrow [Y'] \cap [Z'] \neq \emptyset)$$

From the fact that  $X$  is a non-empty open regular set, we have  $Y' \cup Z' = [Y' \cup Z']^\circ$ . The equivalence follows considering  $Y = Y'$  and  $Z = Z'$ .  $\blacklozenge$

**(DEC)**  $\text{EC}(x, y) =_{\text{def}} \text{C}(x, y) \wedge \neg \text{O}(x, y)$  and  $\llbracket \text{EC}(x, y) \rrbracket_{\alpha-\delta} = [X] \cap [Y] \neq \emptyset \wedge X \cap Y = \emptyset$ .

*Proof*— Trivial.  $\blacklozenge$

## B.2 PROOF OF PROPOSITION 5.1.2

**(DC1)**  $\text{C}_1(x, y) =_{\text{def}} \exists z (\text{S}(z) \wedge \forall z' (\text{CNC}(z', z) \rightarrow (\text{O}(z', x) \wedge \text{O}(z', y))))$  and

$$\llbracket \text{C}_1(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{C}(x, y) \rrbracket = [X] \cap [Y] \neq \emptyset.$$

*Proof*— We have to prove that in all the domains:

$$[X] \cap [Y] \neq \emptyset \iff \exists Z, c, r (Z = \text{ball}(c, r) \wedge \forall Z', r' (Z' = \text{ball}(c, r') \rightarrow (Z' \cap X \neq \emptyset \wedge Z' \cap Y \neq \emptyset))).$$

( $\Rightarrow$ ) It is sufficient to consider  $c \in [X] \cap [Y]$  and  $r > 0$ .

( $\Leftarrow$ ) (By contradiction) Suppose  $[X] \cap [Y] = \emptyset$  and  $Z = \text{ball}(c, r)$ . If  $c \notin \partial(X) \cup \partial(Y)$ , then it is sufficient to consider  $r'$  small enough such that  $Z' = \text{ball}(c, r') \subseteq X$  (if  $c \in X$ ) or  $Z' = \text{ball}(c, r') \subseteq Y$  (if  $c \in Y$ ) or  $Z' = \text{ball}(c, r') \subseteq \mathbb{R}^n - ([X] \cup [Y])$  (if  $c \notin [X] \cup [Y]$ ). If  $c \in \partial(X)$  and  $c \notin [Y]$ , i.e.,  $\text{dist}(c, Y) > 0$ , it suffices to consider  $r'$  small enough such that  $Z' = \text{ball}(c, r') \cap [Y] = \emptyset$ . Analogously for  $c \in \partial(Y)$ . Contradiction.  $\blacklozenge$

**(DSR1)**  $\text{SR}_1(x) =_{\text{def}} \forall y, z (\text{SUM}(x, y, z) \rightarrow \exists s (\text{S}(s) \wedge \text{O}(s, y) \wedge \text{O}(s, z) \wedge \text{P}(s, x)))$  and

$$\llbracket \text{SR}_1(x) \rrbracket_{\alpha-\delta} = \llbracket \text{SR}(x) \rrbracket = \text{Conx}(X).$$

*Proof*— We have to prove that in all the domains:

$$\text{Conx}(X) \iff \forall Y, Z (X = [Y \cup Z]^\circ \rightarrow \exists S, c, r (S = \text{ball}(c, r) \wedge S \cap Y \neq \emptyset \wedge S \cap Z \neq \emptyset \wedge S \subseteq X)).$$

( $\Rightarrow$ ) Since  $Y$  and  $Z$  are regular open, from L.3 and L.4 we have  $X = [Y \cup Z]^\circ \supseteq Y \cup Z$ . From the definition of  $\text{Conx}$ , if  $X = [Y \cup Z]^\circ = Y \cup Z$ , then  $Y \cap Z \neq \emptyset$  and, by L.7,  $Y \cap Z$  is open. Then, there exists a ball  $S$ , such that  $S \subseteq Y \cap Z$ . If  $X = [Y \cup Z]^\circ \supset Y \cup Z$ , then there exists  $x$  such that  $x \in \partial(Y) \cap \partial(Z)$  and  $x \in X$ .  $X$  is open, thus there exists a ball  $S$  centered at  $x$  such that  $S \subseteq X$ ,  $S \cap Y \neq \emptyset$ , and  $S \cap Z \neq \emptyset$ .

( $\Leftarrow$ ) (By contradiction) Assume that there exist  $Y, Z$  such that  $X = Y \cup Z$  and  $Y \cap Z = \emptyset$ . Since  $X$  is regular,  $X = [Y \cup Z]^\circ$ . But  $Y, Z$  are open and disjoint, thus any ball  $S$  intersecting both  $Y$  and  $Z$  intersects  $\mathbb{R}^n - X$  also, then  $\neg S \subseteq X$ . Contradiction.  $\blacklozenge$

## B.3 PROOF OF PROPOSITION 5.1.3

**(DCC1)**  $\text{CCon}_1(z, x, y) =_{\text{def}} \exists z' (\text{CG}(z', z) \wedge \text{C}_1(z', x) \wedge \text{C}_1(z', y))$  and

$$\llbracket \text{CCon}_1(z, x, y) \rrbracket_\alpha = \exists z, z', x, y (z, z' \in [Z] \wedge x \in [X] \wedge y \in [Y] \wedge \text{dist}(x, y) = \text{dist}(z, z')).$$

*Proof* — From Proposition 5.1.2 and the assumption that  $\llbracket \text{CG}(x, y) \rrbracket = \text{Congr}(X, Y)$ , we have to prove that in  $\Phi_\alpha$ :

$$\exists z, z', x, y (z, z' \in [Z], x \in [X], y \in [Y] \wedge \text{dist}(x, y) = \text{dist}(z, z')) \iff \exists Z' (\text{Congr}(Z', Z) \wedge [Z'] \cap [X] \neq \emptyset \wedge [Z'] \cap [Y] \neq \emptyset).$$

( $\Rightarrow$ ) It is sufficient to consider  $Z' = f(Z)$  where  $f$  is an isometry in  $\mathbb{R}^n$  with  $f(z) = x$  and  $f(z') = y$ .

( $\Leftarrow$ ) From the fact that the congruence relation preserves distance.  $\blacklozenge$

#### B.4 PROOF OF PROPOSITION 5.1.4

$$\text{(DS2)} \quad \text{S}_2^*(x) =_{\text{def}} \text{SR}(x) \wedge \forall y, z ((\text{CG}(x, y) \wedge \text{PO}(x, y) \wedge \text{DIF}(z, x, y)) \rightarrow \text{SR}(z)) \text{ and} \\ \llbracket \text{S}_2^*(x) \rrbracket_\beta = \llbracket \text{S}(x) \rrbracket \text{ and } \llbracket \text{S}_2^*(x) \rrbracket_{\alpha, \gamma, \delta} \neq \llbracket \text{S}(x) \rrbracket$$

*Proof* — A counterexample in  $D_\alpha$  is given by a region equal to  $\mathbb{R}^n$  minus a closed ball. This region satisfies the definition but is not a sphere in  $\mathbb{R}^n$ . The definition does not work for  $D_\gamma$  and  $D_\delta$  because in these domains a region  $z$  satisfying  $\text{DIF}(z, x, y)$  must be already a connected region. The proof that the definition above is correct in  $D_\beta$  is quite complicated and it is not reported here.  $\blacklozenge$

#### B.5 PROOF OF PROPOSITION 5.1.6

$$\text{(DC4*)} \quad \text{C}_4^*(x, y) =_{\text{def}} \forall z (\text{CCon}(z, x, y)) \text{ and} \\ \llbracket \text{C}_4^*(x, y) \rrbracket_{\beta, \delta} = \llbracket \text{C}(x, y) \rrbracket = [X] \cap [Y] \neq \emptyset \text{ and } \llbracket \text{C}_4^*(x, y) \rrbracket_{\alpha, \gamma} \neq \llbracket \text{C}(x, y) \rrbracket$$

*Proof* — First, we show that the following equivalence holds in  $\Phi_\beta$  and  $\Phi_\delta$ :

$$[X] \cap [Y] \neq \emptyset \iff \forall Z (\text{dist}(X, Y) \leq \text{diam}(Z))$$

Since we restrict ourselves to domains with finite regions, we can apply L.15 to get:  $[X] \cap [Y] \neq \emptyset$  if and only if  $\text{dist}(X, Y) = 0$ . Note that for all  $\varepsilon > 0$  there exists  $Z$  non-empty such that  $\text{diam}(Z) < \varepsilon$ , then  $\text{dist}(X, Y) = 0$  if and only if  $\forall Z (\text{dist}(X, Y) \leq \text{diam}(Z))$ .

In  $D_\alpha$  and  $D_\gamma$  the interpretation  $[X] \cap [Y] \neq \emptyset$  fails. Consider, for example, regions  $X = \{(a, b) \in \mathbb{R}^2 \mid a > 0 \text{ and } b \geq 1/a\}$  and  $Y = \{(a, b) \in \mathbb{R}^2 \mid b \leq 0\}$ ; it is easy to verify that all the constraints on the domains are satisfied and that  $\text{dist}(X, Y) = 0$  although  $[X] \cap [Y] = \emptyset$ .  $\blacklozenge$

#### B.6 PROOF OF PROPOSITION 5.1.7

$$\text{(DP4*)} \quad \text{P}_4^*(x, y) =_{\text{def}} \forall z (\text{C}_4^*(z, x) \rightarrow \text{C}_4^*(z, y)) \text{ and} \\ \llbracket \text{P}_4^*(x, y) \rrbracket_{\alpha, \delta} = \llbracket \text{P}(x, y) \rrbracket = X \subseteq Y.$$

*Proof* — First, note that in all the domains the interpretation of relation  $\text{C}_4^*(x, y)$  as defined in (DC4\*) is  $\text{dist}(X, Y) = 0$ . This follows from the proof of B.5 dropping the first step through L.15. Thus, we have to prove that in all the domains:

$$X \subseteq Y \iff \forall Z (\text{dist}(Z, X) = 0 \rightarrow \text{dist}(Z, Y) = 0).$$

( $\Rightarrow$ ) Trivial.

( $\Leftarrow$ ) (By contradiction) Suppose  $\neg X \subseteq Y$ . Since  $X$  and  $Y$  are both open, there exists a non-empty regular open set  $V$  such that  $V \subset X$  and  $V \cap Y = \emptyset$ . Let  $Z$  be a non-empty open ball contained in  $V$  such that  $\partial(Z) \cap \partial(V) = \emptyset$ . We have  $\text{dist}(Z, X) = 0$  and  $\text{dist}(Z, Y) \neq 0$ . Contradiction.  $\blacklozenge$

$$\text{(DP4}^+) \quad \text{P}_4^+(x, y) =_{\text{def}} \forall z, w (\text{CCon}(w, z, x) \rightarrow \text{CCon}(w, z, y)) \text{ and} \\ \llbracket \text{P}_4^+(x, y) \rrbracket_{\alpha, \delta} = \llbracket \text{P}(x, y) \rrbracket = X \subseteq Y.$$

*Proof* — It suffices to prove that in all the domains:

$$X \subseteq Y \iff \forall Z, W (dist(Z, X) \leq diam(W) \rightarrow dist(Z, Y) \leq diam(W)).$$

( $\Rightarrow$ ) Directly from the definition of distance.

( $\Leftarrow$ ) (By contradiction) Suppose  $\neg(X \subseteq Y)$ . Since  $X$  and  $Y$  are both open, there exists a non-empty finite open ball  $Z \subset X$  such that  $dist(Z, Y) > 0$ , thus  $Z$  non-tangential to  $X-Y$ . Clearly,  $dist(Z, X) = 0$  and we can always find  $W$  such that  $0 < diam(W) < dist(Z, Y)$ . Contradiction.  $\blacklozenge$

**(DCI4)**  $Closer_4(z, x, y) =_{\text{def}} \exists a (CCon(a, z, x) \wedge \neg CCon(a, z, y))$  and

$$\llbracket Closer_4(z, x, y) \rrbracket_{\alpha-\delta} = \llbracket Closer(z, x, y) \rrbracket = dist(Z, X) < dist(Z, Y).$$

*Proof*— It suffices to prove that in all the domains:

$$dist(Z, X) < dist(Z, Y) \iff \exists A (dist(X, Z) \leq diam(A) \wedge \neg dist(Z, Y) \leq diam(A))$$

This follows from the fact that  $\exists A (dist(X, Z) \leq diam(A) \wedge \neg dist(Z, Y) \leq diam(A))$  is equivalent to  $\exists A (dist(X, Z) \leq diam(A) < dist(Z, Y))$  and that, for all  $\varepsilon > 0$  we can find a non-empty open ball  $A$  such that  $0 < diam(A) < \varepsilon$ .  $\blacklozenge$

## B.7 PROOF OF PROPOSITION 5.1.8

**(DP6)**  $P_6(x, y) =_{\text{def}} \forall z (C(z, x) \rightarrow C(z, y))$  and

$$\llbracket P_6(x, y) \rrbracket_{\alpha-\delta} = \llbracket P(x, y) \rrbracket = X \subseteq Y.$$

*Proof*— From the proof of case (DP) in Proposition 5.1.1.  $\blacklozenge$

**(DCM6)**  $Compl_6(y, x) =_{\text{def}} \forall z (C(z, y) \leftrightarrow \neg IP(z, x))$  and

$$\llbracket Compl_6(y, x) \rrbracket_{\alpha-\delta} = (Y = (\mathbb{R}^n - X)^\circ).$$

*Proof*— Using the results of the Proposition 5.1.1, we have to prove that in all the domains:

$$Y = (\mathbb{R}^n - X)^\circ \iff \forall Z ([Z] \cap [Y] \neq \emptyset \leftrightarrow \neg([Z] \subseteq X)).$$

( $\Rightarrow$ ) If  $Y = (\mathbb{R}^n - X)^\circ$ , then  $[Z] \cap [Y] \neq \emptyset \leftrightarrow [Z] \cap (\mathbb{R}^n - X) \neq \emptyset \leftrightarrow \neg[Z] \subseteq X$ .

( $\Leftarrow$ ) (By contradiction) Assume  $Y \neq (\mathbb{R}^n - X)^\circ$ . We have to prove that (i)  $\exists Z ([Z] \cap [Y] \neq \emptyset \wedge [Z] \subseteq X)$  or (ii)  $\exists Z ([Z] \cap [Y] = \emptyset \wedge \neg([Z] \subseteq X))$ . If  $X \cap Y \neq \emptyset$ , to verify (i) it is sufficient to consider  $[Z] \subseteq X \cap Y$  ( $Z$  exists since  $X$  and  $Y$  are regular open). If  $X \cap Y = \emptyset$ , then  $Y \subset (\mathbb{R}^n - X)^\circ$  and, since both these sets are open, there exists a non-empty open ball  $Z$  such that  $Z \subset (\mathbb{R}^n - X)^\circ$  and  $Z \cap Y = \emptyset$ . Thus, taking  $Z$  small enough, we have  $[Z] \subset (\mathbb{R}^n - X)^\circ - Y$ . Then, condition (ii) is verified.  $\blacklozenge$

**(DSR6)**  $SR_6(x) =_{\text{def}} \forall y, z, w ((SUM(x, y, z) \wedge Compl_6(w, x)) \rightarrow \exists v (SC(v) \wedge O(v, y) \wedge O(v, z) \wedge \neg C(v, w)))$

$$\llbracket SR_6(x) \rrbracket_{\alpha-\delta} = \llbracket SR(x) \rrbracket = Conx(X).$$

*Proof*— (Recall definition (DSC) from Proposition 5.1.1.) We have to prove that in all the domains:

$$Conx(X) \iff \forall Y, Z, W (X = [Y \cup Z]^\circ \wedge W = (\mathbb{R}^n - X)^\circ \rightarrow \exists V (WConx(V) \wedge V \cap Y \neq \emptyset \wedge V \cap Z \neq \emptyset \wedge [V] \cap [W] = \emptyset)).$$

First, since  $X$  is open, we have  $[V] \cap [W] = \emptyset \leftrightarrow [V] \cap (\mathbb{R}^n - X) = \emptyset \leftrightarrow [V] \subseteq X$ .

( $\Rightarrow$ ) Suppose  $X = [Y \cup Z]^\circ$ . From L.3 and L.4, since these are all open non-empty regions, we have  $Y \cup Z \subseteq X$ .

(a) If  $Y \cup Z = X$ , then (from the definition of  $Conx(X)$  and L.7) set  $A = Y \cap Z$  is open, non-empty, and regular. The thesis follows since  $A \subseteq X$  and we can always find a non-empty open ball  $V$  in  $A$  (so that  $WConx(V)$  by L.16, the definition of  $WConx$  and L.2) such that  $[V] \subseteq A$ .

(b) If  $X = [Y \cup Z]^{\circ}$  and  $Y \cup Z \subset X$ , then  $\partial(Y) \cap \partial(Z) \cap X \neq \emptyset$ . Since  $X$  is open, fix a non-empty open ball  $V$  centered in  $p \in \partial(Y) \cap \partial(Z) \cap X$  with  $[V] \subseteq X$ . From the definition of boundary, we have that  $V \cap Y \neq \emptyset \wedge V \cap Z \neq \emptyset$ .

( $\Leftarrow$ ) (By contradiction) Suppose there exist non-empty open regions  $Y$  and  $Z$  such that  $X = Y \cup Z$  and  $Y \cap Z = \emptyset$ . From the regularity of  $X$ ,  $X = Y \cup Z = [Y \cup Z]^{\circ}$ . Fix  $V$  such that  $WConx(V) \wedge V \cap Y \neq \emptyset \wedge V \cap Z \neq \emptyset$ , we prove that  $\neg([V] \subseteq X)$ . Let consider  $VY = V \cap Y$ ,  $VZ = V \cap Z$ , and  $U = VY \cup VZ$  (note that  $VY$  and  $VZ$  not necessarily belong to a domain  $D$  among  $D_{\alpha-\delta}$ , but this does not invalidate our inference because these regions are not used as values of some variable). Clearly  $U$  is in  $V$ .

(a) If  $U \subset V$ , then  $\exists p(p \in V \wedge p \notin VY \wedge p \notin VZ)$ . But  $X = Y \cup Z$  and so  $p \notin X$ . From this,  $V \subseteq X$  is false and, by L.2, it follows that  $\neg([V] \subseteq X)$ .

(b) If  $U = V$ , from the definition of  $WConx(V)$  it follows that  $[VY] \cap [VZ] \neq \emptyset$ . But from  $Y \cap Z = \emptyset$ , we get  $VY \subseteq Y$  and  $VZ \subseteq Z$ , so that  $VY \cap VZ = \emptyset$ . This means that  $\exists p \in \partial(VY) \cap \partial(VZ)$ . Given a neighborhood  $I(p)$  of  $p$ , we have  $I(p) \cap VY \neq \emptyset$  and  $I(p) \cap VZ \neq \emptyset$  and so  $I(p) \cap Y \neq \emptyset$  and  $I(p) \cap Z \neq \emptyset$ . Thus  $p \in \partial(Y) \cap \partial(Z)$ . Since  $Y$  and  $Z$  are open,  $p \notin Y \cup Z$  and, in particular,  $p \notin X$ . Finally, from  $(VY \cup VZ) = V$  and L.3, it follows that  $p \in [V]$  and  $p \notin X$ , i.e.  $\neg[V] \subseteq X$ . Contradiction.  $\blacklozenge$

### B.8 PROOF OF PROPOSITION 5.2.2

(DC3\*)  $C_3^*(x, y) =_{\text{def}} \forall z(\text{Conj}(z, z, x, y))$  and

$$\llbracket C_3^*(x, y) \rrbracket_{\alpha-\delta} = (\text{dist}(X, Y) = 0).$$

*Proof*— We have to prove that in all the domains:

$$\text{dist}(X, Y) = 0 \iff \forall Z \exists z, z', x, y (z, z' \in [Z] \wedge x \in [X] \wedge y \in [Y] \wedge \text{dist}(z, z') = \text{dist}(x, y)).$$

( $\Rightarrow$ ) If  $\text{dist}(X, Y) = 0$ , then for each  $d > 0$ , there exist  $x \in X$  and  $y \in Y$  such that  $\text{dist}(x, y) < d$ . In particular, take a non-empty open ball  $B \subseteq Z$  and consider  $x \in X$  and  $y \in Y$  such that  $\text{dist}(x, y) < \text{diam}(B)$ . From L.13 and L.16, there exist  $z, z' \in B$  such that  $\text{dist}(z, z') = \text{dist}(x, y)$ .

( $\Leftarrow$ ) (By contradiction) Let  $\text{dist}(X, Y) = d > 0$ . It is sufficient to consider  $Z$  such that  $\text{diam}(Z) < d$  to get a contradiction.  $\blacklozenge$

(DP3)  $P_3(x, y) =_{\text{def}} \forall z(C_3^*(z, x) \rightarrow C_3^*(z, y))$  and

$$\llbracket P_3(x, y) \rrbracket_{\alpha-\delta} = \llbracket P(x, y) \rrbracket = X \subseteq Y.$$

*Proof*— This equivalence follows from above and B.6.  $\blacklozenge$

### B.9 PROOF OF PROPOSITION 5.2.3

(DC5\*)  $C_5^*(x, y) =_{\text{def}} \neg \exists z(\text{Closer}(x, z, y))$  and

$$\llbracket C_5^*(x, y) \rrbracket_{\alpha-\delta} = (\text{dist}(X, Y) = 0).$$

*Proof*— We have to prove that in all the domains:

$$\text{dist}(X, Y) = 0 \iff \neg \exists Z(\text{dist}(X, Z) < \text{dist}(X, Y))$$

( $\Rightarrow$ ) Trivial.

( $\Leftarrow$ ) (By contradiction) Let  $\text{dist}(X, Y) > 0$ . It suffices to put  $Z = X$  to reach a contradiction.  $\blacklozenge$

(DP5)  $P_5(x, y) =_{\text{def}} \forall z(C_5^*(z, x) \rightarrow C_5^*(z, y))$  and

$$\llbracket P_5(x, y) \rrbracket_{\alpha-\delta} = \llbracket P(x, y) \rrbracket = X \subseteq Y.$$

*Proof*— It follows from B.6.  $\blacklozenge$

(DFD5)  $FD_5(x) =_{\text{def}} \exists z(\forall x', x''((P_5(x', x) \wedge P_5(x'', x)) \rightarrow \text{Closer}(x', x'', z)))$  and

$$\llbracket \text{FD}_5(x) \rrbracket_{\alpha-\delta} = \text{diam}(X) < +\infty.$$

*Proof*— It suffices to prove the following for all  $X$ :

$$\text{diam}(X) < +\infty \iff \exists Z(\forall X', X''((X' \subseteq X \wedge X'' \subseteq X) \rightarrow \text{dist}(X', X'') < \text{dist}(X', Z))).$$

( $\Rightarrow$ ) If  $\text{diam}(X) < +\infty$ , then consider a non-empty open ball  $Z$  such that  $\text{dist}(Z, X) > \text{diam}(X)$ . (Clearly, such a ball exists in  $\mathbb{R}^n$ .)

( $\Leftarrow$ ) (By contradiction) Assume  $\text{diam}(X) = +\infty$  and let  $\text{dist}(X, Z) = d$ . Since  $\text{diam}(X) = +\infty$ , we can choose a real number  $r$  and two points  $x, y \in X$  such that  $\text{dist}(x, y) > d + 2r$  and  $\text{dist}(x, Z) \leq d + r$ . Let  $X' \subseteq X$  be a ball centered at  $x$  with diameter less or equal to  $r$  and  $X'' \subseteq X$  a ball centered at  $y$  with diameter less or equal to  $r$  as well (these balls exist because  $X$  is open). Then,  $\text{dist}(X', X'') > \text{dist}(X', Z)$ .

Contradiction.  $\blacklozenge$

**(DC5)**  $\text{C}_5(x, y) =_{\text{def}} \exists z, w(\text{FD}_5(z) \wedge \text{FD}_5(w) \wedge \text{P}_5(z, x) \wedge \text{P}_5(w, y) \wedge \text{C}_5^*(z, w))$  and

$$\llbracket \text{C}_5(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{C}(x, y) \rrbracket = [X] \cap [Y] \neq \emptyset.$$

*Proof*— We have to prove the following equivalence in all the domains:

$$[X] \cap [Y] \neq \emptyset \iff \exists Z, W(\text{diam}(Z) < +\infty \wedge \text{diam}(W) < +\infty \wedge Z \subseteq X \wedge W \subseteq Y \wedge \text{dist}(Z, W) = 0).$$

( $\Rightarrow$ ) Fix  $p \in [X] \cap [Y]$ . In  $\Phi_{\alpha, \beta}$ , a region  $Z$  with finite diameter such that  $Z \subseteq X$  and  $p \in [Z]$  is given by  $\text{ball}(p, r) \cap X$ . Analogously for  $W$ . In  $\Phi_{\gamma, \delta}$ , by L.17, there exists  $Z \subseteq X$  such  $\text{diam}(Z) < +\infty$ , and  $p \in [Z]$ . Analogously, for  $W \subseteq Y$ . Since  $p \in [Z] \cap [W]$ , we have  $\text{dist}(Z, W) = 0$ .

( $\Leftarrow$ ) From the hypothesis we have that  $\text{diam}(Z) < +\infty \wedge \text{diam}(W) < +\infty \wedge \text{dist}(Z, W) = 0$ . From L.15, we have  $[Z] \cap [W] \neq \emptyset$ . But,  $Z \subseteq X$  and  $W \subseteq Y$ , thus  $[X] \cap [Y] \neq \emptyset$ .  $\blacklozenge$

## APPENDIX C

### C.1 PROOF OF LEMMA 5.3.1

The interpretations of MSP and  $\Sigma$ SS are obtained by substituting the interpretations of the components in the definition. For this reason there is nothing to prove and, in the attempt to improve readability, we simply write  $\llbracket \text{MSP} \rrbracket(X, Y)$  and  $\llbracket \Sigma \text{SS} \rrbracket(X)$  for the interpretations of MSP( $x, y$ ) and  $\Sigma$ SS( $x$ ), respectively. Given this premise, we prove that:

$$\text{(A.1)} \quad \llbracket \text{SCG}(x, y) \rrbracket_{\alpha-\delta} = \exists c_x, c_y, r (X = \text{Ball}(c_x, r) \wedge Y = \text{Ball}(c_y, r));$$

$$\text{(A.2)} \quad \llbracket \text{EqD}(x, y, x', y') \rrbracket_{\alpha-\delta} = \exists c_x, c_y, c'_x, c'_y, r, r' (X = \text{Ball}(c_x, r) \wedge Y = \text{Ball}(c_y, r') \wedge X' = \text{Ball}(c'_x, r) \wedge Y' = \text{Ball}(c'_y, r') \wedge \neg X \subseteq Y \wedge \neg Y \subseteq X \wedge \neg X' \subseteq Y' \wedge \neg Y' \subseteq X' \wedge \text{dist}(c_x, c_y) = \text{dist}(c'_x, c'_y));$$

$$\text{(A.3)} \quad \llbracket \Sigma \text{CG}(x, y) \rrbracket_{\alpha-\delta} = \text{Congr}(X, Y) \wedge \llbracket \Sigma \text{SS} \rrbracket(X) \wedge \llbracket \Sigma \text{SS} \rrbracket(Y);$$

$$\text{(A)} \quad \llbracket \text{CG}_1(x, y) \rrbracket_{\alpha-\delta} = \text{Congr}(X, Y).$$

*Proof*— **(A.1)** Let  $X = \text{Ball}(c_x, r_x)$  and  $Y = \text{Ball}(c_y, r_y)$ . We have to prove the following equivalence in all the domains:

$$\text{Congr}(X, Y) \iff X = Y \vee \exists c, r_z, r_w (Z = \text{Ball}(c, r_z) \wedge W = \text{Ball}(c, r_w) \wedge [Z] \cap [X] \neq \emptyset \wedge Z \cap X = \emptyset \wedge [Z] \cap [Y] \neq \emptyset \wedge Z \cap Y = \emptyset \wedge X \subset W \wedge \partial(X) \cap \partial(W) \neq \emptyset \wedge Y \subset W \wedge \partial(Y) \cap \partial(W) \neq \emptyset).$$

( $\Rightarrow$ ) From  $r_x = r_y$ , one has  $\text{diam}(X) = \text{diam}(Y)$ . If  $X = Y$  we are done. Otherwise  $c_x \neq c_y$  and  $X, Y$  properly overlap or are disjoint. Fix  $c \notin [X \cup Y]$  such that  $\text{dist}(c_x, c) = \text{dist}(c_y, c) = r$ , and let  $Z = \text{Ball}(c, r - r_x)$ . By construction,  $[Z] \cap [X] \neq \emptyset \wedge Z \cap X = \emptyset \wedge [Z] \cap [Y] \neq \emptyset \wedge Z \cap Y = \emptyset$ . Let  $W = \text{Ball}(c, r + r_x)$ . By construction,  $X, Y \subset W$  and  $\partial(X) \cap \partial(W) \neq \emptyset, \partial(Y) \cap \partial(W) \neq \emptyset$ .

( $\Leftarrow$ ) If  $X = Y$  we are done. If not, from the hypothesis  $X, Y \subset (W - Z)$ ,  $[Z] \cap [X] \neq \emptyset$ ,  $[Z] \cap [Y] \neq \emptyset$  and  $\partial(X) \cap \partial(W) \neq \emptyset$ ,  $\partial(Y) \cap \partial(W) \neq \emptyset$ . By triangular inequality, the minimum distance between a point in  $\partial(W)$  and a point in  $\partial(Z)$  is  $r_w - r_z$ . Then  $r_w - r_z \leq \text{diam}(X)$ . Analogously,  $r_w - r_z \leq \text{diam}(Y)$ . Assume  $r_w - r_z < \text{diam}(X)$ . From  $Z \cap X = \emptyset$ , there exists a point in  $X$  such that its distance from  $c$  is higher than  $r_w$ . Contradiction. Then  $r_w - r_z = \text{diam}(X)$ . Analogously,  $r_w - r_z = \text{diam}(Y)$ . Finally,  $\text{diam}(X) = \text{diam}(Y)$ .  $\blacklozenge$

**(A.2)** Let  $X = \text{Ball}(c_x, r)$ ,  $Y = \text{Ball}(c_y, r')$ ,  $X' = \text{Ball}(c'_x, r')$ , and  $Y' = \text{Ball}(c'_y, r')$  such that  $\neg X \subseteq Y \wedge \neg Y \subseteq X \wedge \neg X' \subseteq Y' \wedge \neg Y' \subseteq X'$ . We have to prove the following equivalence in all the domains:

$$\text{dist}(c_x, c_y) = \text{dist}(c'_x, c'_y) \iff \exists c_z, c_w, r'' (Z = \text{Ball}(c_z, r'') \wedge W = \text{Ball}(c_w, r'') \wedge \text{IntD}(z, x, y) \wedge \text{IntD}(w, x', y'))$$

( $\Rightarrow$ ) The minimum sphere  $S$  containing  $X$  and  $Y$  has diameter  $\text{dist}(c_x, c_y) + r + r'$ . Analogously,  $\text{dist}(c'_x, c'_y) + r + r'$  is the radius of the minimal sphere  $S'$  containing  $X'$  and  $Y'$ . Thus,  $S$  and  $S'$  are congruent. Also, by triangular inequality, the center of  $S$  is between  $c_x, c_y$ , i.e.,  $X$  and  $Y$  are internally diametrical to  $S$ . Analogously for  $S'$ .

( $\Leftarrow$ ) Since  $Z$  has  $X$  and  $Y$  as internally diametrical spheres, then the center of  $Z$  is between the centers of  $X$  and  $Y$ , so  $r'' = \text{dist}(c_x, c_y) + r + r'$ . Analogously, for  $W$ . Then,  $\text{dist}(c_x, c_y) = \text{dist}(c'_x, c'_y)$ .  $\blacklozenge$

**(A.3)** Assume  $\llbracket \Sigma SS \rrbracket(X)$  and  $\llbracket \Sigma SS \rrbracket(Y)$ , then we have to prove the following equivalence in all the domains:

$$\begin{aligned} \text{Congr}(X, Y) \iff & \forall S (\llbracket \text{MSP} \rrbracket(S, X) \rightarrow \exists S' (\llbracket \text{MSP} \rrbracket(S', Y) \wedge \text{Congr}(S, S'))) \wedge \\ & \forall S (\llbracket \text{MSP} \rrbracket(S, Y) \rightarrow \exists S' (\llbracket \text{MSP} \rrbracket(S', X) \wedge \text{Congr}(S, S'))) \wedge \\ & \forall S, U, S', U' (\llbracket \text{MSP} \rrbracket(S, X) \wedge \llbracket \text{MSP} \rrbracket(U, X) \wedge \llbracket \text{MSP} \rrbracket(S', Y) \wedge \llbracket \text{MSP} \rrbracket(U', Y) \wedge \\ & \text{Congr}(S, S') \wedge \text{Congr}(U, U')) \rightarrow \llbracket \text{EqD} \rrbracket(S, U, S', U') \end{aligned}$$

( $\Rightarrow$ ) Let  $f$  be an isometry such that  $Y = f(X)$  and put  $S' = f(S)$  for  $S$  in  $X$ ,  $S' = f^{-1}(S)$  for  $S$  in  $Y$ . The conditions follow easily.

( $\Leftarrow$ ) We show that there exists an isometry  $f$  such that  $Y = f(X)$ . We write  $c_i$  for the center of sphere  $S_i$ , and  $c'_i$  for the center of sphere  $S'_i$ . For each pair  $S_i, S_j$  of maximal spheres  $S$  in  $X$ , let  $x_{ij}, x_{ji}$  be the points on the boundary of  $S_i, S_j$ , respectively, such that  $\text{dist}(x_{ij}, x_{ji}) = \text{diam}(S_i \cup S_j)$ . Let  $c_1, \dots, x_{mn}, \dots$  be a list of all centers of maximal spheres in  $X$  and of all points isolated above. We show that there exists an isometry  $f$  such that  $c'_i = f(c_i)$  and  $\text{dist}(x, y) = \text{dist}(f(x), f(y))$  for any pair of points  $x, y$  in the list and that  $S' = f(S)$  for each maximal sphere  $S$  in  $X$  (and so  $S' = f^{-1}(S)$  for  $S$  in  $Y$ ). From the first two conditions in the hypothesis, we can choose  $f$  such that  $\text{dist}(x, y) = \text{dist}(f(x), f(y))$  for  $x, y$  centers of maximal spheres in  $X$ . Relation  $\text{dist}(x_{ij}, x_{ji}) = \text{dist}(f(x_{ij}), f(x_{ji}))$  can also be satisfied because of the third constraint in the hypothesis. The other pairs follow from triangular inequality. From Euclidian geometry, we know that this function can be extended to an isometry on the whole space. Since the center of a maximal sphere  $S$  is mapped to the center of its congruent sphere  $S'$  (and so the points in the boundary where  $f$  is constrained, if any) and  $f$  is an isometry, we also have  $S' = f(S)$ . We conclude that  $Y = f(X)$ .  $\blacklozenge$

**(A)** We have to prove the following equivalence in all the domains:

$$\begin{aligned} \text{Congr}(X, Y) \iff & \forall Z (\llbracket \Sigma SS \rrbracket(Z) \rightarrow \exists Z' (\text{Congr}(Z, Z') \wedge \\ & \forall S, S' (\llbracket \text{MSP} \rrbracket(S, Z) \wedge \llbracket \text{MSP} \rrbracket(S', Z') \wedge \text{Congr}(S, S')) \rightarrow \\ & ((S \subseteq X \iff S' \subseteq Y) \wedge (S \subseteq Y \iff S' \subseteq X) \wedge \\ & ((S \cap X \neq \emptyset \wedge \neg S \subseteq X \wedge \neg X \subseteq S) \iff (S' \cap Y \neq \emptyset \wedge \neg S' \subseteq Y \wedge \neg Y \subseteq S)) \wedge \\ & ((S' \cap X \neq \emptyset \wedge \neg S' \subseteq X \wedge \neg X \subseteq S') \iff (S \cap Y \neq \emptyset \wedge \neg S \subseteq Y \wedge \neg Y \subseteq S)))) \end{aligned}$$

( $\Rightarrow$ ) Let  $f$  be an isometry such that  $Y = f(X)$ . Fix any  $Z$  such that  $\llbracket \Sigma SS \rrbracket(Z)$  holds and let  $Z' = f(Z)$ . It is easy (although tedious) to verify the conditions since  $f$  is an isometry.

( $\Leftarrow$ ) The proof splits in four cases. Let  $n$  be the dimension of the space.

Case (I) If  $X$  is the whole space then so is  $Y$ . If not, it suffices to take a  $Z$  that partially overlaps  $Y$ . Analogously if  $Y$  is the whole space.

Case (II)  $\llbracket \Sigma SS \rrbracket(X)$  and the convex hull of the centers of the maximal spheres in  $X$  is a region of dimension  $n$ . Put  $Z = X$  and let  $Z'$  be as in the hypothesis. Let  $f$  be the isometry such that  $Z' = f(Z)$ . We show that  $Z' = Y$ . For this, it suffices to show that  $Z' \neq Y$  leads to a contradiction. Choose  $n$  maximal spheres of  $X$  such that the convex hull of their centers is a region of dimension  $n$ . Call  $W$  the sum of these spheres. Let  $W' = f(W)$ . Since  $\llbracket \Sigma SS \rrbracket(X)$  and the hypothesis on the maximal spheres of  $X$ , one must have  $Z' \subseteq Y$ . Let  $Y - Z' \neq \emptyset$ . Since the regions  $Y$  and  $Z'$  are regular and open, one can find a ball  $U' \subseteq Y - Z'$  such that no maximal sphere in  $W'$  has the diameter of  $U'$  and all maximal spheres of  $W'$  are maximal in  $W' \cup U'$  as well. Furthermore, we take  $U'$  such that  $Z' \cap U' = \emptyset$  in structures  $\Phi_\alpha$  and  $\Phi_\beta$ . In structures  $\Phi_\gamma$  and  $\Phi_\delta$ , we also add a new region  $C' \subseteq Y$  that connects  $W'$  and  $U'$  (this condition is necessary to guarantee that region  $Z'_U$ , which we are going to construct, exists in these structures).

Let  $Z'_U = Z' \cup U'$  (we use  $U' \cup C'$  instead of  $U'$  in structures  $\Phi_\gamma$  and  $\Phi_\delta$ ) and let  $U = f^{-1}(U')$ . Fix  $Z_U = Z \cup U$ , i.e.,  $Z'_U = f(Z_U)$ . Since  $Z' \subset Z'_U$  and  $f$  is an isometry,  $Z \subset Z_U$ . Thus,  $X \subset Z_U$ . Now, apply the hypothesis to  $Z'_U$  to get a region  $Z''_U$ . Since  $Z'_U \subseteq Y$ , one must have  $Z''_U \subseteq X$ . By construction and the choices of  $Z$  and of  $U$ ,  $Z'$  is congruent to  $X$  and  $Z'_U$  is congruent to  $Z''_U$ , thus  $Z'_U$  is congruent to  $Z'$  (or a part of  $Z'$ ). But  $Z' \subset Z'_U$ . Contradiction. We conclude  $Z' = Y$ , that is,  $\text{Congr}(X, Y)$ .

Case (III)  $\llbracket \Sigma SS \rrbracket(X)$  and the convex hull of the centers of the maximal spheres in  $X$  is a region of dimension less than  $n$ .

We proceed as before but this time region  $W$  must contain some sphere which is disjoint from  $X$ . (Again, in structures  $\Phi_\gamma$  and  $\Phi_\delta$  we also consider a region  $C$  that connects  $W'$  and  $Y$  in such a way that the maximal spheres of  $Y$  and of  $W'$  do not change. One gets the conclusion as in Case (II) by considering the isometry  $f$  identified by  $Z' = f(Z)$ ).

Case (IV) Not  $\llbracket \Sigma SS \rrbracket(X)$ .

Since  $X$  is not the whole space, there exists a sequence  $Z_i$ , with  $\llbracket \Sigma SS \rrbracket(Z_i)$ ,  $X \subset Z_i$ , and  $\cup Z_{i+1} \subset \cup Z_i$ , that converges to  $X$ . Let  $Z'_i$  be the region satisfying the hypothesis when applied to  $Z_i$  and let  $f_i$  be the function for which  $Z'_i = f_i(Z_i)$ . Since  $\llbracket \Sigma SS \rrbracket(Z_i)$ ,  $f_i$  must be an isometry. Furthermore,  $f_{i+1}(\cup Z_{i+1}) \subset f_i(\cup Z_i)$ ,  $f_i(X) \subset f_i(\cup Z_i)$ , and, for  $i \rightarrow +\infty$ ,  $\text{Vol}(\cup Z_i) \rightarrow \text{Vol}(X)$ . From these, the sequence  $Z'_i$  converges to a region (call it  $Z'$ ) containing  $Y$ . First we show that  $\text{Congr}(X, Z')$ . Assume not, then there exists a set  $N$  of points in  $X$  (or in  $Z'$ ) such that  $f(N) \not\subset Z'$  for all isometries  $f$  (analogously  $f(N) \not\subset X$  for all isometries  $f$ , if  $N$  in  $Z'$ ). Note that  $N \subset X \subset Z_i$  for all  $i$ . From  $f(N) \not\subset Z'$  and the fact that  $Z'$  is the limit of  $f_i(Z_i)$ , there is an index  $m$  such that  $f_m(N) \not\subset Z'_m$ . Let  $m$  be the first index for which this happens. By construction  $f_m$  is an isometry and  $f_m(X) = Z'_m$  contradicting  $f_m(N) \not\subset Z'_m$ . We have seen that  $Y \subseteq Z'$  and  $\text{Congr}(X, Z')$ . It remains to show that  $Z' - Y \neq \emptyset$  leads to a contradiction. This follows from considering a new sequence  $W_i$ , with  $\llbracket \Sigma SS \rrbracket(W_i)$ ,  $W_i \subset X$ , and  $W_i \subset W_{i+1}$ , that converges to  $X$  and the sequence defined by  $f(W_i) = W'_i$  for all  $i$ , since  $f(W_i) \subset Y$  for all  $i$ .  $\blacklozenge$

## C.2 PROOF OF LEMMA 5.3.2

We prove that:

- (A.1)  $\llbracket C_2^*(x, y) \rrbracket_{\alpha-\delta} = (\text{dist}(X, Y) = 0)$ ;
- (A.2)  $\llbracket SC_2^*(x) \rrbracket_{\alpha-\delta} = WW\text{Conx}(X)$ ;
- (A.3)  $\llbracket LEDiam_2(x, y) \rrbracket_{\alpha-\delta} = WW\text{Conx}(Y) \wedge \text{diam}(X) \leq \text{diam}(Y)$ ;
- (A.4)  $\llbracket SCDiam_2(x, y) \rrbracket_{\alpha-\delta} = WW\text{Conx}(X) \wedge \text{diam}(X) \leq \text{diam}(Y)$ ;

(A.5)  $\llbracket \text{LDist}_2(x, y, x', y') \rrbracket_{\alpha-\delta} = \text{dist}(X, Y) < \text{dist}(X', Y')$ ;

(A)  $\llbracket \text{CCon}_2(c, x, y) \rrbracket_{\alpha-\delta} = \text{dist}(X, Y) \leq \text{diam}(C)$ .

*Proof*— (A.1) We have to prove the following equivalence:

$$\text{dist}(X, Y) = 0 \iff \forall Z \exists Z' (\text{Congr}(Z', Z) \wedge Z' \cap X \neq \emptyset \wedge Z' \cap Y \neq \emptyset).$$

( $\Rightarrow$ ) If  $\text{dist}(X, Y) = 0$ , then for each  $d > 0$  there exist  $x \in X$  and  $y \in Y$  such that  $\text{dist}(x, y) < d$ . In particular, take a ball  $B \subseteq Z$  and consider  $x$  and  $y$  such that  $\text{dist}(x, y) < \text{diam}(B)$ . From L.13, there exist  $z, z' \in B$  such that  $\text{dist}(z, z') = \text{dist}(x, y)$ . It suffices to consider an isometry  $f$  such that  $Z' = f(Z)$ ,  $x = f(z)$  and  $y = f(z')$ .

( $\Leftarrow$ ) (By contradiction) Assume  $\text{dist}(X, Y) = d > 0$  and consider  $Z$  such that  $\text{diam}(Z) < d$ . Clearly, the hypothesis fails.  $\blacklozenge$

(A.2) Directly from, (A.1), Proposition 5.1.1, and the definition of  $WW\text{Conx}$ .  $\blacklozenge$

(A.3) For every w-weakly connected region  $Y$ , we have to prove the following equivalence in all the domains:

$$\text{diam}(X) \leq \text{diam}(Y) \iff \forall A, B (A \subseteq X \wedge B \subseteq Y \rightarrow \exists Y' (\text{Congr}(Y', Y) \wedge Y' \cap A \neq \emptyset \wedge Y' \cap B \neq \emptyset)).$$

( $\Rightarrow$ ) Let  $a \in A$  and  $b \in B$  be two points in  $X$ . Since  $\text{diam}(X) \leq \text{diam}(Y)$  and  $Y$  is w-weak connected, from L.18 there exist  $y, y' \in Y$  such that  $\text{dist}(y, y') = \text{dist}(a, b)$ . It is enough to consider an isometry  $f$  with  $a = f(y)$  and  $b = f(y')$  and to take  $Y' = f(Y)$ .

( $\Leftarrow$ ) (By contradiction) Let us assume  $\text{diam}(X) > \text{diam}(Y)$ . It follows that there exist  $a, b \in X$  with  $\text{dist}(a, b) - \text{diam}(Y) = \varepsilon > 0$ . Let  $A$  and  $B$  be two balls in  $X$  with diameter smaller than  $\varepsilon/2$  (such  $A$  and  $B$  exist because  $X$  is an open region). Clearly we have  $A \subseteq X$ ,  $B \subseteq X$ , and  $\text{dist}(A, B) > \text{diam}(Y)$ . Contradiction.  $\blacklozenge$

(A.4) For every w-weakly connect region  $X$ , we have to prove the following equivalence in all the domains:

$$\text{diam}(X) \leq \text{diam}(Y) \iff \forall Z ((Y \subseteq Z \wedge WW\text{Conx}(Z)) \rightarrow \text{diam}(X) \leq \text{diam}(Z)).$$

( $\Rightarrow$ ) Since  $Y \subseteq Z$ , we have  $\text{diam}(Y) \leq \text{diam}(Z)$ , thus (from the hypothesis)  $\text{diam}(X) \leq \text{diam}(Z)$ .

( $\Leftarrow$ ) Let  $Z$  be the *convex-hull* of  $Y$ . By the definition of the convex-hull,  $Y \subseteq Z$  and, by L.19,  $\text{diam}(Z) = \text{diam}(Y)$ . By L.16, we have  $\text{Conx}(Z)$  and, from L.1,  $WW\text{Conx}(Z)$ . It follows that  $\text{diam}(X) \leq \text{diam}(Y)$ .  $\blacklozenge$

(A.5) We have to prove the following equivalence in all the domains:

$$\begin{aligned} \text{dist}(X, Y) < \text{dist}(X', Y') &\iff \\ \exists A (WW\text{Conx}(A) \wedge A \cap X \neq \emptyset \wedge A \cap Y \neq \emptyset \wedge \forall A' (\text{Congr}(A', A) \rightarrow (A' \cap X' = \emptyset \vee A' \cap Y' = \emptyset))) & \end{aligned}$$

( $\Rightarrow$ ) It is enough to consider a ball  $A$  overlapping both  $X$  and  $Y$  with diameter equal to  $\text{dist}(X, Y) + \varepsilon$ , where  $\varepsilon < \text{dist}(X', Y') - \text{dist}(X, Y)$ .

( $\Leftarrow$ ) (By contradiction) Let us assume  $\text{dist}(X, Y) \geq \text{dist}(X', Y')$ . Let  $A$  be an arbitrary open region such that  $WW\text{Conx}(A)$  and  $x \in \{A \cap X\}$ ,  $y \in \{A \cap Y\}$ . Since  $\text{dist}(x, y) \geq \text{dist}(X', Y')$  there are  $x' \in X'$ ,  $y' \in Y'$  such that  $\text{dist}(x', y') \leq \text{dist}(x, y)$ . From L.18, there exist  $a, a' \in A$  such that  $\text{dist}(a, a') = \text{dist}(x', y')$ . We obtain a contradiction taking  $A' = f(A)$  with  $f$  an isometry such that  $a = f(x')$  and  $a' = f(y')$ .  $\blacklozenge$

(A) We have to prove the following equivalence in all the domains:

$$\text{dist}(X, Y) \leq \text{diam}(C) \iff$$

$$\forall A, B (\text{dist}(A, B) < \text{dist}(X, Y) \rightarrow \exists Z (W\text{Conx}(Z) \wedge \text{diam}(Z) \leq \text{diam}(C) \wedge Z \cap A \neq \emptyset \wedge Z \cap B \neq \emptyset))$$

( $\Rightarrow$ ) For  $C$  infinite, take  $Z = \mathbb{R}^n$ . Let now  $C$  be finite, i.e.,  $\text{diam}(C) < +\infty$ . Let  $A$  and  $B$  be such that  $\text{dist}(A, B) < \text{dist}(X, Y)$ . From the proof of (A.5), we have  $\exists Z (WW\text{Conx}(Z) \wedge Z \cap A \neq \emptyset \wedge Z \cap B \neq \emptyset \wedge \forall Z' (\text{Congr}(Z', Z) \rightarrow (Z' \cap X = \emptyset \vee Z' \cap Y = \emptyset)))$ . Since  $\text{Congr}(Z', Z) \rightarrow \text{diam}(Z') = \text{diam}(Z)$ , from  $\forall Z' (\text{Congr}(Z', Z) \rightarrow (Z' \cap X = \emptyset \vee Z' \cap Y = \emptyset))$ , we have  $\text{diam}(Z) \leq \text{dist}(X, Y)$ , then, from the hypothesis,  $\text{diam}(Z) \leq \text{diam}(C)$ . From L.1 and  $\text{diam}(C) < +\infty$ ,  $W\text{Conx}(Z)$ .

( $\Leftarrow$ ) (By contradiction) Let us assume  $\text{dist}(X, Y) > \text{diam}(C)$ . Let  $A \subseteq X$  and  $B \subseteq Y$  be two open regions with  $\text{dist}(A, B) - \text{diam}(C) = \varepsilon > 0$ . From the definition of  $\text{dist}$  and  $\text{diam}$ , if  $Z$  is such that  $\text{diam}(Z) \leq \text{diam}(C)$ , then  $Z$  cannot overlap both  $A$  and  $B$ . Contradiction.  $\blacklozenge$

### C.3 PROOF OF LEMMA 5.3.4

Using (DP) in Proposition 5.1.1, and (DFD5) in section 5.2, we need to prove that:

$$(A) \quad \llbracket \text{Conj}_\delta(x, y, x', y') \rrbracket_{\alpha-\delta} = \llbracket \text{Conj}(x, y, x', y') \rrbracket$$

We start proving that:

$$(A.1) \quad \llbracket \text{Eq}(z, x, y) \rrbracket_{\alpha-\delta} = (\text{dist}(Z, X) = \text{dist}(Z, Y));$$

$$(A.2) \quad \llbracket \text{EqD}^*(x, y, x', y') \rrbracket_{\alpha-\delta} = (\text{dist}(X, Y) = \text{dist}(X', Y')) \wedge$$

$$\text{diam}(X) < +\infty \wedge \text{diam}(Y) < +\infty \wedge \text{diam}(X') < +\infty \wedge \text{diam}(Y') < +\infty.$$

*Proof* — (A.1) It holds because of the following (obvious) equivalence

$$\text{dist}(Z, X) = \text{dist}(Z, Y) \iff \neg \text{dist}(Z, X) < \text{dist}(Z, Y) \wedge \neg \text{dist}(Z, Y) < \text{dist}(Z, X). \quad \blacklozenge(A.1)$$

Regarding (A.2), the two alternative definitions (a) and (b) of  $\text{EqD}^*$  need to be considered:

(A.2.a) Following the definition (a) of  $\text{EqD}^*$ , assuming  $X, Y, X', Y'$  of finite diameter, we need to prove that in  $\Phi_\alpha$  and  $\Phi_\beta$  for  $\mathbb{R}^1$  and in  $\Phi_{\alpha-\delta}$  for  $\mathbb{R}^{n>1}$  the following equivalence holds:

$$\text{dist}(X, Y) = \text{dist}(X', Y') \iff \exists Z, Z' (\text{dist}(X, Y) = \text{dist}(X, Z) \wedge \text{dist}(X', Y') = \text{dist}(X', Z') \wedge \text{dist}(Z, X) = \text{dist}(Z, Z') \wedge \text{dist}(Z', X') = \text{dist}(Z', Z))$$

( $\Rightarrow$ ) (Sketch) Put  $\text{dist}(X, Y) = \text{dist}(X', Y') = d$ .

(a) Assume  $d = 0$ . The thesis follows considering a ball  $Z = Z'$  containing  $X \cup X'$ .

(b) Assume  $d > 0$  and  $n = 1$ . Suppose that there exists  $p \in \partial(X)$  such that all the points in  $X \cup X'$  lie on the same side of  $\mathbb{R}^1$  with respect to  $p$  (a similar argument holds for  $p' \in \partial(X')$  such that all the points in  $X \cup X'$  lie on the same side of  $\mathbb{R}^1$  with respect to  $p$ ). Let  $Z$  be finite and connected such that  $\text{dist}(Z, X) = d$  with  $Z$  on the opposite side of  $X$  with respect to  $p$ . Let  $Z'_1$  be finite and connected such that  $\text{dist}(Z'_1, Z) = d$  with  $Z'_1$  on the opposite side of  $p$  with respect to  $Z$ . Let  $Z'_2$  be finite connected such that  $\text{dist}(Z'_2, X') = d$  with  $Z'_2$  on the opposite side of  $p$  with respect to  $X'$ . By construction  $\text{dist}(Z'_1, Z'_2) \geq d$ . Then it suffices to put  $Z' = Z'_1 \cup Z'_2$  (since  $Z'$  is not connected, this proof holds only in  $\Phi_\alpha$  and  $\Phi_\beta$ ).

(c) Assume  $d > 0$  and  $n > 1$ . Consider  $X^\wedge = \{x \in \mathbb{R}^n \mid \text{dist}(x, X) < d\}$ ,  $X'^\wedge = \{x \in \mathbb{R}^n \mid \text{dist}(x, X') < d\}$ .  $X^\wedge$  and  $X'^\wedge$  have finite diameter, therefore  $\mathbb{R}^n - [X^\wedge \cup X'^\wedge]$  has at most one connected component with infinite diameter. Call it  $V$ . Let  $p \in \partial(V) \cap \partial(X'^\wedge)$  (otherwise take  $p \in \partial(V) \cap \partial(X^\wedge)$  and switch  $X'^\wedge, X^\wedge$  in the rest of the proof). By construction, there exists  $q \in \partial(X^\wedge)$  such that  $q$  is path-connected to  $p$  in  $\mathbb{R}^n - X^\wedge$  and  $\text{dist}(p, q) > d$ . Let  $Z'$  be a connected region in  $V$  such that  $p \in \partial(Z')$  and  $\text{diam}(Z') < (d - \text{dist}(p, q))/2$ . Let  $Z$  be a connected region in  $\mathbb{R}^n - X^\wedge$ , such that  $q \in \partial(Z)$  and  $\text{dist}(Z, Z') = d$  (since  $q$  is path-connected

to  $p$  in  $\mathbb{R}^n - X^\wedge$  and  $dist(p, q) > d$  this region exists always). Clearly we have  $dist(Z, X) = dist(Z, Z') = dist(Z', X') = d$ .

( $\Leftarrow$ ) Trivial.

**(A.2.b)** In  $\Phi_{\gamma,\delta}$  for  $\mathbb{R}^1$ , we use the definition **(b)** of  $\text{EqD}^*$ . First we prove that:

**(B.1)**  $\llbracket \text{CG}^s(x, y) \rrbracket_{\gamma,\delta} = (diam(X) = diam(Y) < +\infty \wedge [X] \cap [Y] = \emptyset)$ ;

**(B.2)**  $\llbracket \text{CG}^*(x, y) \rrbracket_{\gamma,\delta} = (diam(X) = diam(Y) < +\infty)$ .

**(B.1)** Given two connected regions  $X, Y$  with finite diameter (in  $\Phi_{\gamma,\delta}$  all the regions are *Conx*) such that  $[X] \cap [Y] = \emptyset$ , we need to prove the following equivalence:

$$diam(X) = diam(Y) \iff \exists Z_1, Z_2, Z_3 (([Z_2] \cap [X] \neq \emptyset \wedge Z_2 \cap X = \emptyset \wedge [Z_2] \cap [Y] \neq \emptyset \wedge Z_2 \cap Y = \emptyset \wedge [Z_1] \cap [X] \neq \emptyset \wedge Z_1 \cap X = \emptyset \wedge [Z_1] \cap [Z_2] = \emptyset \wedge [Z_3] \cap [Y] \neq \emptyset \wedge Z_3 \cap Y = \emptyset \wedge [Z_3] \cap [Z_2] = \emptyset \wedge dist(Z_2, Z_1) = dist(Z_2, Z_3))$$

From the hypothesis  $X$  and  $Y$  are connected regions and are not connected to each other. Let  $X = (x_1, x_2)$ ,  $Y = (y_1, y_2)$ , and assume that  $x_2$  strictly precedes  $y_1$  (a similar argument holds if we take  $y_2$  precedes  $x_1$ ). Since  $Z_2$  needs to be externally connected to both  $X$  and  $Y$  and  $[X] \cap [Y] = \emptyset$ , then  $Z_2 = (x_2, y_1)$ .  $Z_1$  ( $Z_3$ ) needs to be externally connected to  $X$  ( $Y$ ) and it does not overlap  $Z_2$ , therefore  $Z_1 = (x_1 - d_1, x_1)$  ( $Z_3 = (y_2, y_2 + d_2)$ ) for some  $d_1 < +\infty$  ( $d_2 < +\infty$ ). By construction,  $dist(Z_2, Z_1) = diam(X)$  and  $dist(Z_2, Z_3) = diam(Y)$ .

◆**(B.1)**

**(B.2)** Given two connected regions  $X, Y$  (in  $\Phi_{\gamma,\delta}$  all the regions are *Conx*) with finite diameter, we need to prove the following equivalence:

$$diam(X) = diam(Y) \iff$$

(a)  $X = Y \vee$

(b)  $(X \cap Y \neq \emptyset \wedge \neg X \subseteq Y \wedge \neg Y \subseteq X \wedge \exists Z_1, Z_2 (Z_1 = X - [Y] \wedge Z_2 = Y - [X] \wedge [Z_1] \cap [Z_2] = \emptyset \wedge diam(Z_1) = diam(Z_2))) \vee$

(c)  $([X] \cap [Y] \neq \emptyset \wedge X \cap Y = \emptyset \wedge \neg \exists Z ((Z \subset X \wedge diam(Z) = diam(Y) \wedge [Z] \cap [Y] = \emptyset) \vee (Z \subset Y \wedge diam(Z) = diam(X) \wedge [Z] \cap [X] = \emptyset))) \vee$

(d)  $([X] \cap [Y] = \emptyset \wedge diam(X) = diam(Y))$ .

On the basis of the *parthood* and *connection* relations there are a total of eight distinct cases to consider between two connected regions: (1)  $\partial(X) \cap \partial(Y) \neq \emptyset$  and  $X \subset Y$ , (2)  $\partial(X) \cap \partial(Y) \neq \emptyset$  and  $Y \subset X$ , (3)  $\partial(X) \cap \partial(Y) = \emptyset$  and  $X \subset Y$ , (4)  $\partial(X) \cap \partial(Y) = \emptyset$  and  $Y \subset X$ , (5)  $X = Y$ , (6)  $X$  and  $Y$  partially overlap, (7)  $X$  and  $Y$  are externally connected, and (8)  $X$  and  $Y$  are not connected. In cases (1)-(4), the regions have necessarily different diameters since we are in  $\mathbb{R}^1$ . In addition, conditions (a)–(d) correspond to cases (5)–(8), thus we can exclude cases (1)-(4) altogether.

Let  $X = (x_1, x_2)$ ,  $Y = (y_1, y_2)$ .

Case (5): both  $diam(X) = diam(Y)$  and condition (a) follows.

Case (6): we have  $X \cap Y \neq \emptyset$   $\neg X \subseteq Y \wedge \neg Y \subseteq X$  and since  $X$  and  $Y$  are connected and finite, then also  $Z_1 = X - [Y]$  and  $Z_2 = Y - [X]$  are connected and finite. In addition, by definition, we have  $[Z_1] \cap [Z_2] = \emptyset$ . In this case  $diam(X) = diam(Z_1) + diam(X \cap Y)$  and  $diam(Y) = diam(Z_2) + diam(X \cap Y)$ , i.e.  $diam(X) = diam(Y)$  iff  $diam(Z_1) = diam(Z_2)$ .

Case (7): we have  $[X] \cap [Y] \neq \emptyset \wedge X \cap Y = \emptyset$ , therefore there exists  $x$  such that  $[X] \cap [Y] = \{x\}$ . If  $diam(X) = diam(Y)$ , condition (c) is easily verified. For the other direction, condition (c) holds  $diam(Z) < diam(Y)$  for all  $Z \subset X$ . Thus,  $diam(X) \leq diam(Y)$ . Analogously, one shows  $diam(Y) \leq diam(X)$ .

Case (8): we have  $[X] \cap [Y] = \emptyset$ . Then,  $diam(X) = diam(Y)$  and condition (d) are equivalent.  $\blacklozenge(\mathbf{B.2})$

Following the definition **(b)** of  $\text{EqD}^*$ , assuming  $X, Y, X', Y'$  of finite diameter and connected, we have now to prove that in  $\Phi_{\alpha, \delta}$  for  $\mathbb{R}^1$  the following equivalence holds:

$$\begin{aligned} dist(X, Y) = dist(X', Y') \iff & ([X] \cap [Y] \neq \emptyset \wedge [X'] \cap [Y'] \neq \emptyset) \vee \exists Z, Z' ([Z] \cap [X] \neq \emptyset \wedge Z \cap X = \emptyset \wedge \\ & [Z] \cap [Y] \neq \emptyset \wedge Z \cap Y = \emptyset \wedge [Z'] \cap [X'] \neq \emptyset \wedge Z' \cap X' = \emptyset \wedge [Z'] \cap [Y'] \neq \emptyset \wedge Z' \cap Y' = \emptyset \wedge \\ & diam(Z) = diam(Z')) \end{aligned}$$

( $\Rightarrow$ ) Let  $dist(X, Y) = dist(X', Y')$ . If  $[X] \cap [Y] \neq \emptyset$  and  $[X'] \cap [Y'] \neq \emptyset$ , we are done. Assume  $[X] \cap [Y] = \emptyset$ . From L.15,  $dist(X, Y) > 0$  and so  $dist(X', Y') > 0$ , i.e.,  $[X'] \cap [Y'] = \emptyset$ . Let  $Z = \{z \in \mathbb{R}^1 \mid z \notin [X \cup Y] \text{ and } Btw(z, x, y) \text{ for some } x \in X, y \in Y\}$  and  $Z' = \{z' \in \mathbb{R}^1 \mid z' \notin [X' \cup Y'] \text{ and } Btw(z', x', y') \text{ for some } x' \in X', y' \in Y'\}$ . Thus,  $diam(Z) = dist(X, Y) = dist(X', Y') = diam(Z')$  and the other conditions are satisfied by construction.

( $\Leftarrow$ ) (By contradiction) Let  $dist(X, Y) > dist(X', Y')$  so that  $[X] \cap [Y] = \emptyset$ . If there exist  $Z$  and  $Z'$  externally tangent to  $X, Y$  and to  $X', Y'$  (respectively) such that  $diam(Z) = diam(Z')$ , then, since  $Z$  is connected,  $diam(Z) = dist(X, Y)$  and  $diam(Z') = dist(X', Y')$ . But  $dist(X, Y) > dist(X', Y')$  and  $diam(Z) = diam(Z')$ . Contradiction.  $\blacklozenge(\mathbf{A.2.b})/\blacklozenge(\mathbf{A.2})$

We now prove that:

$$(\mathbf{A}) \quad \llbracket \text{Conj}_\delta(x, y, x', y') \rrbracket_{\alpha-\delta} = \llbracket \text{Conj}(x, y, x', y') \rrbracket$$

We begin with a lemma:

**Lemma C.2.1** —  $\forall X, Y, Z, W (([X] \cap [Y] \neq \emptyset \wedge [Z] \cap [W] \neq \emptyset \wedge diam(X) < +\infty \wedge diam(Z) < +\infty) \rightarrow \exists x, z (x \in [X] \cap [Y] \wedge z \in [Z] \cap [W] \wedge dist(x, z) = dist([X] \cap [Y], [Z] \cap [W])))$

*Proof* —  $diam(X) < +\infty$  implies  $diam([X] \cap [Y]) < +\infty$ . Furthermore,  $\llbracket [X] \cap [Y] \rrbracket = [X] \cap [Y]$ . Similarly for  $[Z] \cap [W]$ . The thesis follows from L.12.  $\blacklozenge(\mathbf{Lemma})$

(**A**) It remains to prove the following equivalence that we state somewhat informally in the attempt to improve readability:

$$\exists x, y, x', y' (x \in [X] \wedge y \in [Y] \wedge x' \in [X'] \wedge y' \in [Y'] \wedge dist(x, y) = dist(x', y')) \iff$$

There exist four finite and connected regions  $A, B, A', B'$  (the finiteness of the diameter follows from the  $\text{EqD}^*$  condition) with properties:

$$[A] \cap [X] \neq \emptyset, [B] \cap [Y] \neq \emptyset, [A'] \cap [X'] \neq \emptyset, [B'] \cap [Y'] \neq \emptyset, dist(A, B) = dist(A', B');$$

and such that:

1.  $\forall P_A, P_B ((P_A \subseteq A \wedge P_B \subseteq B \wedge dist(P_A, P_B) = dist(A, B)) \rightarrow [P_A] \cap [X] \neq \emptyset \wedge [P_B] \cap [Y] \neq \emptyset)$
2.  $\forall P_{A'}, P_{B'} ((P_{A'} \subseteq A' \wedge P_{B'} \subseteq B' \wedge dist(P_{A'}, P_{B'}) = dist(A', B')) \rightarrow [P_{A'}] \cap [X'] \neq \emptyset \wedge [P_{B'}] \cap [Y'] \neq \emptyset).$

( $\Rightarrow$ ) *Case*  $n = 1$ . Without loss of generality, assume  $x \leq y$  and fix a value  $r < +\infty$ . Consider the open balls  $A = (x - r, x)$  and  $B = (y, y + r)$ . Clearly,  $x \in [A] \cap [X], y \in [B] \cap [Y]$  and condition 1. is satisfied since  $x$  must belong to  $[P_A]$  and  $y$  to  $[P_B]$ . We can build  $A'$  and  $B'$  analogously from  $x', y'$ . Then,  $dist(A, B) = dist(A', B'), [A'] \cap [X'] \neq \emptyset, [B'] \cap [Y'] \neq \emptyset$ , and condition 2. is satisfied as well.

*Case*  $n > 1$ . Let  $r$  be finite.

(a) Assume  $x = y$  (and so  $x' = y'$ ), then it suffices to consider  $A = ball(x, r) \cap X, B = ball(x, r) \cap Y, A' = ball(x', r) \cap X'$ , and  $B' = ball(x', r) \cap Y'$  (or appropriate connected subregions of these).

(b) Assume  $x \neq y$  (and so  $x' \neq y'$ ).

Fix the line  $l$  through  $x$  and  $y$  and two finite balls  $A$  and  $B$  with centers in  $l$  and such that  $x \in \partial(A)$ ,  $y \in x \in \partial(B)$ , and  $\text{dist}(A, B) = \text{dist}(x, y)$ . Analogously, find  $A'$  and  $B'$  with centers in  $l'$  (the line through  $x'$  and  $y'$ ),  $x' \in [A']$ ,  $y' \in [B']$ , and  $\text{dist}(A', B') = \text{dist}(x', y')$ . By construction, we have  $[A] \cap [X] \neq \emptyset$ ,  $[B] \cap [Y] \neq \emptyset$ ,  $[A'] \cap [X'] \neq \emptyset$ ,  $[B'] \cap [Y'] \neq \emptyset$ ,  $\text{dist}(A, B) = \text{dist}(A', B')$ . Regarding conditions 1. and 2., observe that only the two pairs  $x, y$  and  $x', y'$ , (in  $A, B$  and in  $A', B'$ , respectively) have distance equal to  $\text{dist}(A, B)$ . Thus, if  $P_A \subseteq A \wedge P_B \subseteq B \wedge \text{dist}(P_A, P_B) = \text{dist}(A, B)$  and  $P_{A'} \subseteq A' \wedge P_{B'} \subseteq B' \wedge \text{dist}(P_{A'}, P_{B'}) = \text{dist}(A', B')$ , we must have  $x \in [P_A]$ ,  $y \in [P_B]$ ,  $x' \in [P_{A'}]$ , and  $y' \in [P_{B'}]$ . This guarantees that conditions 1. and 2. are satisfied.

( $\Leftarrow$ ) From the hypothesis  $A, B, A'$ , and  $B'$  have finite diameter. Also,  $[A] \cap [X] \neq \emptyset$  and  $[B] \cap [Y] \neq \emptyset$ . We show that  $\text{dist}([A] \cap [X], [B] \cap [Y]) = \text{dist}(A, B)$ . From the definition of  $\text{dist}$ ,  $\text{dist}([A] \cap [X], [B] \cap [Y]) \geq \text{dist}(A, B)$ . Suppose  $\text{dist}([A] \cap [X], [B] \cap [Y]) > \text{dist}(A, B)$ . From L.12, there exist  $a \in [A]$  and  $b \in [B]$  such that  $\text{dist}(a, b) = \text{dist}(A, B)$ . Let  $d = \text{dist}([A] \cap [X], [B] \cap [Y]) - \text{dist}(A, B)$ ,  $S_A = A \cap \text{ball}(a, d/3)$ , and  $S_B = B \cap \text{ball}(b, d/3)$  (or appropriate connected subregions). Since  $\text{diam}(S_A \cup S_B) \leq \text{dist}(A, B) + 2/3d < \text{dist}([A] \cap [X], [B] \cap [Y])$ , we have  $[S_A] \cap [X] = \emptyset$  or  $[S_B] \cap [Y] = \emptyset$ . This contradicts condition 1., thus  $\text{dist}([A] \cap [X], [B] \cap [Y]) = \text{dist}(A, B)$ . Analogously, we have  $\text{dist}([A'] \cap [X'], [B'] \cap [Y']) = \text{dist}(A', B')$ . From  $\text{dist}(A, B) = \text{dist}(A', B')$ , one obtains  $\text{dist}([A] \cap [X], [B] \cap [Y]) = \text{dist}([A'] \cap [X'], [B'] \cap [Y'])$ . From Lemma C.2.1, there exist  $x, y, x'$ , and  $y'$  such that  $x \in [A] \cap [X]$ ,  $y \in [B] \cap [Y]$ ,  $x' \in [A'] \cap [X']$ ,  $y' \in [B'] \cap [Y']$  and  $\text{dist}(x, y) = \text{dist}(x', y') = \text{dist}(A, B)$ .  $\blacklozenge$

#### C.4 PROOF OF LEMMA 5.3.5

We start proving that:

$$(A.1) \quad \llbracket \text{FD}_3(x) \rrbracket_{\alpha-\delta} = \text{diam}(X) < +\infty.$$

$$(A) \quad \llbracket \text{C}_3(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{C}(x, y) \rrbracket.$$

*Proof* — (A.1) First, we need to find the interpretation of  $\text{SC}_3^*$ . From the proof of Proposition 5.2.2 (see Appendix B.8) it follows that  $\llbracket \text{C}_3^*(x, y) \rrbracket_{\alpha-\delta} = \text{dist}(X, Y) = 0$ . From Proposition 5.1.1 and the definition of  $WW\text{Conx}$  we directly obtain that  $\llbracket \text{SC}_3^*(x, y) \rrbracket_{\alpha-\delta} = WW\text{Conx}(X)$ . Thus, we need to prove the following formula in all the domains:

$$\text{diam}(X) < +\infty \iff$$

$$\exists X', Y, Z (WW\text{conx}(X') \wedge X \subseteq X' \wedge \neg \exists x, x', y, z (x, x' \in [X'] \wedge y \in [Y] \wedge z \in [Z] \wedge \text{dist}(x, x') = \text{dist}(y, z))).$$

( $\Rightarrow$ ) If  $d = \text{diam}(X) < +\infty$ , it is sufficient to consider three balls  $X', Y$ , and  $Z$  of finite diameter  $d' \geq d$  such  $X \subseteq X'$  and  $\text{dist}(Y, Z) > d'$ .

( $\Leftarrow$ ) (By contradiction) Let  $\text{diam}(X) = +\infty$  and  $WW\text{conx}(X') \wedge X \subseteq X'$ . Clearly  $\text{diam}(X') = +\infty$ . From L.18, for all  $d \in \mathbb{R}$ , there exist  $x, x' \in [X']$  such that  $\text{dist}(x, x') = d$ . Fix any  $Y$  and  $Z$ . Then, for all  $y \in [Y]$  and  $z \in [Z]$ , we can find  $x, x' \in [X']$  such that  $\text{dist}(x, x') = \text{dist}(y, z)$ . Contradiction.  $\blacklozenge$

(A) Let us observe that  $\llbracket \text{FD}_3(x) \rrbracket_{\alpha-\delta} = \llbracket \text{FD}_5(x) \rrbracket_{\alpha-\delta}$ ,  $\llbracket \text{P}_5(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{P}(x, y) \rrbracket$ ,  $\llbracket \text{C}_3^*(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{C}_5^*(x, y) \rrbracket_{\alpha-\delta}$ , and that the definition of  $\text{C}_3$  is analogous to the one of  $\text{C}_5$ . It is possible to consider the same proof given for  $\text{C}_5$  (see (DC5) in section 5.2) in Appendix B.9.  $\blacklozenge$

Regarding  $\text{S}_3$ , we use the results above and prove that:

$$(B.1) \quad \llbracket \text{LEDiam}_3(x, y) \rrbracket_{\alpha-\delta} = \text{Conx}(X) \wedge \text{Conx}(Y) \wedge \text{diam}(X) \leq \text{diam}(Y);$$

$$(B) \quad \llbracket \text{S}_3(x) \rrbracket_{\alpha-\delta} = \llbracket \text{S}(x) \rrbracket = \exists c, r (X = \text{Ball}(c, r)).$$

*Proof* — **(B.1)** Given two connected open regions  $X$  and  $Y$ , we have to prove the following equivalence in all the domains:

$$\text{diam}(X) \leq \text{diam}(Y) \iff \forall A, B ((A \subseteq X \wedge B \subseteq X) \rightarrow \exists A', B' (A' \subseteq Y \wedge B' \subseteq Y \wedge \exists a, b, a', b' (a \in [A] \wedge b \in [B] \wedge a' \in [A'] \wedge b' \in [B'] \wedge \text{dist}(a, b) = \text{dist}(a', b'))))$$

( $\Rightarrow$ ) Since  $A, B$  are subsets of  $X$  and  $A', B'$  are subsets of  $Y$  it is enough to show that  $\forall a, b (a, b \in [X] \rightarrow \exists a', b' (a', b' \in [Y] \wedge \text{dist}(a, b) = \text{dist}(a', b'))$ . The latter is an immediate consequence of the hypothesis, L.1, and L.13.

( $\Leftarrow$ ) (By contradiction) Assume there exist two connected regions  $X, Y$  such that  $\text{diam}(X) > \text{diam}(Y)$ . By the definition of  $\text{diam}$ , there exist  $a, b \in [X]$  such that for all  $a', b' \in [Y]$ ,  $\text{dist}(a, b) > \text{dist}(a', b')$ . Let  $3d < \text{dist}(a, b) - \text{dist}(a', b')$ , we reach a contradiction considering  $A = X \cap \text{ball}(a, d)$ ,  $B = X \cap \text{ball}(a', d)$  (or some connected parts of them in the case of domains with only connected regions).  $\blacklozenge$

*Proof* — **(B)** We have to prove the following equivalence in all the domains:

$$\begin{aligned} \exists c, r (X = \text{Ball}(c, r)) &\iff \text{diam}(X) < +\infty \wedge \text{Conx}(X) \wedge \forall A (\text{Conx}(A) \wedge \text{diam}(A) \leq \text{diam}(X) \rightarrow \\ &\exists B (\text{Conx}(B) \wedge \text{diam}(B) \leq \text{diam}(A) \wedge B \subset X \wedge \forall C, D ([C] \cap [X] \neq \emptyset \wedge C \cap X = \emptyset \wedge [D] \cap [X] \neq \emptyset \wedge \\ &D \cap X = \emptyset) \rightarrow \exists x, y, x', y' (x \in [B] \wedge y \in [C] \wedge x' \in [B] \wedge y' \in [D] \wedge \text{dist}(x, y) = \text{dist}(x', y'))) \end{aligned}$$

( $\Rightarrow$ ) Given a region  $A$  of diameter  $l$ , it suffices to take a ball  $B$  of diameter  $l$  (or less) concentric to  $X$ . Since the distance of the boundary of  $B$  to a region externally connected to  $X$  is always  $r - l$ , the result follows for every  $C, D$ .

( $\Leftarrow$ ) We proceed by contradiction. Let  $X$  be connected with finite diameter. Recall that if there exists a point  $x$  equidistant to any point in  $\partial(X)$ ,  $X$  is a sphere with center  $x$ . Assume that  $X$  is not a sphere, i.e., for each point  $x$  of  $X$  there are at least two points  $y, z \in \partial(X)$  such that  $\text{dist}(x, y) \neq \text{dist}(x, z)$ .

Let  $F(x) = \max_{y, z \in \partial(X)} (|\text{dist}(x, y) - \text{dist}(x, z)|)$  and fix  $a \in X$  such that  $F(a)$  is minimum, i.e., for all  $x \in X$ ,  $F(a) \leq F(x)$ . Of course,  $F(a) > 0$  since  $[X]$  is compact and is not a sphere. Also, let  $c, d \in \partial(X)$  be such that  $F(a) = |\text{dist}(a, c) - \text{dist}(a, d)|$ . (Again,  $c, d$  exist since  $\partial(X)$  is compact). Now, choose a region  $A$  with  $\text{diam}(A) < F(a)/2$  and fix any  $B \subset X$  as in the hypothesis. Fix  $x$  in  $B$ . By construction, there exist two regions  $C, D$  with  $c \in C$ ,  $d \in D$ , that are externally connected to  $X$  and such that  $|\text{dist}(x, C) - \text{dist}(x, D)| \geq F(a)$ . Since  $\text{diam}(B) < F(a)/2$ , for any point  $y \in B$ ,  $\text{dist}(x, C) + F(a)/4 \geq \text{dist}(y, C) \geq \text{dist}(x, C) - F(a)/4$ . Then, for any  $y, z \in B$ ,  $|\text{dist}(y, C) - \text{dist}(z, D)| \geq F(a) - F(a)/2 > 0$ . Contradiction.  $\blacklozenge$  **(B)**

### C.5 PROOF OF LEMMA 5.3.6

As before, we rely on [56] for the definition of the between relation **BTW**. Note that we write  $\text{Btw}(c_1, c_2, c_3)$  for ‘ $c_1$  is between  $c_2$  and  $c_3$ ’ which corresponds to  $\text{Btw}(c_2, c_1, c_3)$  in Tarski’s terminology. Using (DO), (D+), (DPP) and the results in sections 5.1 and 5.2, it remains to prove that:

$$\text{(A.1)} \quad \llbracket \text{Conv}(x) \rrbracket_{\alpha-\delta} = \text{Conv}(X);$$

$$\text{(A)} \quad \llbracket \text{ConvH}_1(x, y) \rrbracket_{\alpha-\delta} = \llbracket \text{ConvH}(x, y) \rrbracket = \text{Conv}(X) \wedge Y \subseteq X \wedge \neg \exists Z (\text{Conv}(Z) \wedge Y \subseteq Z \wedge Z \subset X)$$

*Proof* — **(A.1)** We prove the following equivalence for  $\text{Conv}(x)$ :

$$\forall x, y, z (x, y \in X \wedge \text{Btw}(z, x, y)) \rightarrow z \in X \iff \forall S_1, S_2, S_3, c_1, c_2, c_3, r_1, r_2, r_3 (S_1 = \text{ball}(c_1, r_1) \wedge S_2 = \text{ball}(c_2, r_2) \wedge S_3 = \text{ball}(c_3, r_3) \wedge S_1 \cup S_2 \subseteq X \wedge \text{Btw}(c_3, c_1, c_2)) \rightarrow S_3 \cap X \neq \emptyset;$$

the proof of (A) follows trivially.

( $\Rightarrow$ ) Consider  $S_1 = ball(c_1, r_1)$ ,  $S_2 = ball(c_2, r_2)$ ,  $S_3 = ball(c_3, r_3)$ ,  $S_1 \cup S_2 \subseteq X$ , and  $Btw(c_3, c_1, c_2)$ . Since  $S_1 \cup S_2 \subseteq X$ , then  $c_1, c_2 \in X$  and, from the hypothesis,  $c_3 \in X$ . Then,  $S_3 \cap X \neq \emptyset$ .

( $\Leftarrow$ ) (By contradiction) Suppose there exist  $x \in X$ ,  $y \in X$ , and  $z \notin X$  such that  $Btw(z, x, y)$ . Thus,  $x \neq z \neq y \neq x$ . We show that  $\exists x', y', z' (Btw(z', x', y') \wedge x', y' \in X \wedge z' \notin [X])$ . From this, the contradiction follows taking the radius of  $S_3$  to be less than  $dist(z', X)$ .

The only case to consider is  $z \in \partial(X)$ . Since  $X$  is open, there exist three congruent balls  $S_x = ball(x, r)$ ,  $S_y = ball(y, r)$ , and  $S_z = ball(z, r)$ , such that  $S_x \subseteq X$ ,  $S_y \subseteq X$ ,  $\neg S_z \subseteq [X]$ ,  $S_x \cap S_y = \emptyset$ ,  $S_x \cap S_z = \emptyset$ , and  $S_y \cap S_z = \emptyset$ . Fix a point  $z'$  in  $S_z - [X]$  and call  $l$  the line through  $x, y, z$ , and  $l'$  line through  $z'$  and parallel to  $l$ . Consider  $x' \in S_x \cap l'$  and  $y' \in S_y \cap l'$ . Since  $S_x, S_y$ , and  $S_z$  are disjoint congruent balls with aligned centers, from  $Btw(z, x, y)$  we conclude  $Btw(z', x', y')$ . Finally, fix three balls  $S_1 = ball(x', r_1)$ ,  $S_2 = ball(y', r_2)$ , and  $S_3 = ball(z', r_3)$ , such that  $S_1 \subseteq X$ ,  $S_2 \subseteq X$ ,  $\neg S_3 \subseteq [X]$  and  $S_1 \cap S_2 = \emptyset$ . This is possible since  $X$  is open and  $z \notin [X]$ .  $\blacklozenge$

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## TABLE OF DEFINITIONS

$C_1(x, y) =_{\text{def}} \exists z(S(z) \wedge \forall z'(CNC(z', z) \rightarrow (O(z', x) \wedge O(z', y))))$
$C_2(x, y) =_{\text{def}} \exists z(S_2^*(z) \wedge \forall z'(CNC(z', z) \rightarrow (O(z', x) \wedge O(z', y))))$
$C_2^*(x, y) =_{\text{def}} \forall z \exists z'(CG(z', z) \wedge O(z', x) \wedge O(z', y))$
$C_3(x, y) =_{\text{def}} \exists z, w(FD_3(z) \wedge FD_3(w) \wedge P(z, x) \wedge P(w, y) \wedge C_3^*(z, w))$
$C_3^*(x, y) =_{\text{def}} \forall z(\text{Conj}(z, z, x, y))$
$C_4^*(x, y) =_{\text{def}} \forall z(\text{CCon}(z, x, y))$
$C_5(x, y) =_{\text{def}} \exists z, w(FD_5(z) \wedge FD_5(w) \wedge P_5(z, x) \wedge P_5(w, y) \wedge C_5^*(z, w))$
$C_5^*(x, y) =_{\text{def}} \neg \exists z(\text{Closer}(x, z, y))$
$\text{CCon}_1(z, x, y) =_{\text{def}} \exists z'(CG(z', z) \wedge C_1(z', x) \wedge C_1(z', y))$
$\text{CCon}_2(c, x, y) =_{\text{def}} \forall a, b(\text{LDist}_2(a, b, x, y) \rightarrow \exists z(\text{SCDiam}_2(z, c) \wedge O(z, a) \wedge O(z, b)))$
$\text{CG}_1(x, y) =_{\text{def}} \forall z(\Sigma SS(z) \rightarrow \exists z'(\Sigma CG(z, z') \wedge \forall s, s'((\text{MSP}(s, z) \wedge \text{MSP}(s', z') \wedge \text{SCG}(s, s')) \rightarrow ((P(s, x) \leftrightarrow P(s', y)) \wedge (P(s, y) \leftrightarrow P(s', x)) \wedge (PO(s, x) \leftrightarrow PO(s', y)) \wedge (PO(s, y) \leftrightarrow PO(s', x))))))$
$\text{CG}^*(x, y) =_{\text{def}} FD_5(x) \wedge FD_5(y) \wedge$ $(x = y \vee$ $(PO(x, y) \wedge \exists z_1, z_2(\text{DIF}(z_1, x, y) \wedge \text{DIF}(z_2, y, x) \wedge \text{CG}^s(z_1, z_2))) \vee$ $(\text{EC}(x, y) \wedge \neg \exists z((PP(z, x) \wedge \text{CG}^s(z, y)) \vee (PP(z, y) \wedge \text{CG}^s(z, x)))) \vee$ $(\neg C(x, y) \wedge \text{CG}^s(x, y)))$
$\text{CG}^s(x, y) =_{\text{def}} FD_5(x) \wedge FD_5(y) \wedge \neg C(x, y) \wedge \exists z_1, z_2, z_3(\text{EC}(z_2, x) \wedge \text{EC}(z_2, y) \wedge \text{EC}(z_1, x) \wedge \neg C(z_1, z_2) \wedge \text{EC}(z_3, y) \wedge \neg C(z_3, z_2) \wedge \text{Eq}(z_2, z_1, z_3))$
$\text{Closer}_4(z, x, y) =_{\text{def}} \exists a(\text{CCon}(a, z, x) \wedge \neg \text{CCon}(a, z, y))$
$\text{Compl}_6(y, x) =_{\text{def}} \forall z(C(z, y) \leftrightarrow \neg IP(z, x))$
$\text{Conj}_5(x, y, x', y') =_{\text{def}} \exists a, b, a', b'(\text{SR}(a) \wedge \text{SR}(b) \wedge \text{SR}(a') \wedge \text{SR}(b') \wedge$ $C(a, x) \wedge C(b, y) \wedge C(a', x') \wedge C(b', y') \wedge \text{EqD}^*(a, b, a', b') \wedge$ $\forall p_a, p_b((P(p_a, a) \wedge P(p_b, b) \wedge \text{EqD}^*(p_a, p_b, a, b)) \rightarrow (C(p_a, x) \wedge C(p_b, y))) \wedge$ $\forall p'_a, p'_b((P(p'_a, a') \wedge P(p'_b, b') \wedge \text{EqD}^*(p'_a, p'_b, a', b')) \rightarrow (C(p'_a, x') \wedge C(p'_b, y')))$
$\text{Conv}(x) =_{\text{def}} \forall s_1, s_2, s_3((P(s_1, x) \wedge P(s_2, x) \wedge \text{BTW}(s_3, s_1, s_2)) \rightarrow O(s_3, x))$
$\text{ConvH}_1(x, y) =_{\text{def}} \text{Conv}(x) \wedge P(y, x) \wedge \neg \exists z(\text{Conv}(z) \wedge P(y, z) \wedge \text{PP}(z, x))$
$\text{DIF}(z, x, y) =_{\text{def}} \forall w(P(w, z) \leftrightarrow (P(w, x) \wedge \neg O(w, y)))$
$\text{EC}(x, y) =_{\text{def}} C(x, y) \wedge \neg O(x, y)$
$\text{Eq}(z, x, y) =_{\text{def}} \neg \text{Closer}(z, x, y) \wedge \neg \text{Closer}(z, y, x)$
$\text{EqD}(x, y, x', y') =_{\text{def}} \text{SCG}(x, x') \wedge \text{SCG}(y, y') \wedge \neg P(x, y) \wedge \neg P(y, x) \wedge \neg P(x', y') \wedge \neg P(y', x') \wedge$ $\exists z, w(\text{ID}(z, x, y) \wedge \text{ID}(w, x', y') \wedge \text{SCG}(z, w))$
$\text{EqD}^*(x, y, x', y') =_{\text{def}} FD_5(x) \wedge FD_5(y) \wedge FD_5(x') \wedge FD_5(y') \wedge$ $\exists z, z'(\text{Eq}(x, y, z) \wedge \text{Eq}(x', y', z') \wedge \text{Eq}(z, x, z') \wedge \text{Eq}(z', x', z))$ (in $\Phi_{\alpha-\beta}$ for $R^1$ and in $\Phi_{\alpha-\delta}$ for $R^{n>1}$ )
$\text{EqD}^*(x, y, x', y') =_{\text{def}} (C(x, y) \wedge C(x', y')) \vee \exists z, z'(\text{EC}(z, x) \wedge \text{EC}(z, y) \wedge \text{EC}(z', x') \wedge \text{EC}(z', y') \wedge \text{CG}^*(z, z'))$ (in $\Phi_{\gamma-\delta}$ for $R^1$ )
$\text{FD}_3(x) =_{\text{def}} \exists x', y, z(\text{SC}_3^*(x') \wedge P(x, x') \wedge \neg \text{Conj}(x', x', y, z))$
$\text{FD}_5(x) =_{\text{def}} \exists z(\forall x', x''((P_5(x', x) \wedge P_5(x'', x)) \rightarrow \text{Closer}(x', x'', z)))$
$\text{IP}(x, y) =_{\text{def}} P(x, y) \wedge \forall z(C(z, x) \rightarrow O(z, y))$
$\text{LDist}_2(x, y, x', y') =_{\text{def}} \exists a(\text{SC}_2^*(a) \wedge O(a, x) \wedge O(a, y) \wedge \forall a'(CG(a', a) \rightarrow (\neg O(a', x') \vee \neg O(a', y'))))$
$\text{LEDiam}_2(x, y) =_{\text{def}} \text{SC}_2^*(y) \wedge \forall a, b((P(a, x) \wedge P(b, x)) \rightarrow \exists y'(CG(y', y) \wedge O(y', a) \wedge O(y', b)))$
$\text{LEDiam}_3(x, y) =_{\text{def}} \text{SR}(x) \wedge \text{SR}(y) \wedge \forall a, b((P(a, x) \wedge P(b, x)) \rightarrow \exists a', b'(P(a', y) \wedge P(b', y) \wedge \text{Conj}(a, b, a', b')))$
$\text{MSP}(x, y) =_{\text{def}} S(x) \wedge P(x, y) \wedge \forall z((S(z) \wedge \text{PP}(x, z)) \rightarrow \neg P(z, y))$

$O(x, y) =_{\text{def}} \exists z(P(z, x) \wedge P(z, y))$
$P^*(x, y) =_{\text{def}} \forall w(C(x, w) \rightarrow C(y, w))$
$P_3(x, y) =_{\text{def}} \forall z(C_3^*(z, x) \rightarrow C_3^*(z, y))$
$P_4^*(x, y) =_{\text{def}} \forall z(C_4^*(z, x) \rightarrow C_4^*(z, y))$
$P_4^+(x, y) =_{\text{def}} \forall z, w(CCon(w, z, x) \rightarrow CCon(w, z, y))$
$P_5(x, y) =_{\text{def}} \forall z(C_5^*(z, x) \rightarrow C_5^*(z, y))$
$P_6(x, y) =_{\text{def}} \forall z(C(z, x) \rightarrow C(z, y))$
$PO(x, y) =_{\text{def}} O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$
$PP(x, y) =_{\text{def}} P(x, y) \wedge \neg P(y, x)$
$S_2^*(x) =_{\text{def}} SR(x) \wedge \forall y, z((CG(x, y) \wedge PO(x, y) \wedge DIF(z, x, y)) \rightarrow SR(z))$
$S_3(x) =_{\text{def}} FD_3(x) \wedge SR(x) \wedge \forall a(LEDiam_3(a, x) \rightarrow \exists b(LEDiam_3(b, a) \wedge P(b, x) \wedge \forall c, d(EC(c, x) \wedge EC(d, x)) \rightarrow (Conj(b, c, b, d)))$
$SC(x) =_{\text{def}} \forall y, z(SUM(x, y, z) \rightarrow C(y, z))$
$SC_2^*(x) =_{\text{def}} \forall y, z(SUM(x, y, z) \rightarrow C_2^*(y, z))$
$SC_3^*(x) =_{\text{def}} \forall y, z(SUM(x, y, z) \rightarrow C_3^*(y, z))$
$SCDiam_2(x, y) =_{\text{def}} SC_2^*(x) \wedge \forall z((P(y, z) \wedge SC_2^*(z)) \rightarrow LEDiam_2(x, z))$
$SCG(x, y) =_{\text{def}} S(x) \wedge S(y) \wedge (x=y \vee \exists z, w(CNC(z, w) \wedge EC(z, x) \wedge EC(z, y) \wedge TPP(x, w) \wedge TPP(y, w)))$
$SR_1(x) =_{\text{def}} \forall y, z(SUM(x, y, z) \rightarrow \exists s(S(s) \wedge O(s, y) \wedge O(s, z) \wedge P(s, x)))$
$SR_6(x) =_{\text{def}} \forall y, z, w((SUM(x, y, z) \wedge Compl_6(w, x)) \rightarrow \exists v(SC(v) \wedge O(v, y) \wedge O(v, z) \wedge \neg C(v, w)))$
$SUM(z, x, y) =_{\text{def}} \forall w(O(w, z) \leftrightarrow (O(w, x) \vee O(w, y)))$
$TPP(x, y) =_{\text{def}} PP(x, y) \wedge \exists z(EC(z, x) \wedge EC(z, y))$
$\Sigma CG(x, y) =_{\text{def}} \Sigma SS(x) \wedge \Sigma SS(y) \wedge \forall s(MSP(s, x) \rightarrow \exists s'(MSP(s', y) \wedge SCG(s, s'))) \wedge \forall s(MSP(s, y) \rightarrow \exists s'(MSP(s', x) \wedge SCG(s, s'))) \wedge \forall s, u, s', u(MSP(s, x) \wedge MSP(u, x) \wedge MSP(s', y) \wedge MSP(u', y) \wedge SCG(s, s') \wedge SCG(u, u')) \rightarrow EqD(s, u, s', u')$
$\Sigma SS(x) =_{\text{def}} \forall y(P(y, x) \rightarrow \exists s(MSP(s, x) \wedge O(s, y))) \wedge \forall u, w((MSP(u, x) \wedge MSP(w, x) \wedge u \neq w) \rightarrow \neg SCG(u, w))$